Darboux transforms on band matrices, weights and associated polynomials

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Abstract

Classically, it is well known that a single weight on a real interval leads to orthogonal polynomials. In "Generalized orthogonal polynomials, discrete KP and Riemann-Hilbert problems", Comm. Math. Phys. 207, pp. 589-620 (1999), we have shown that m-periodic sequences of weights lead to "moments", polynomials defined by determinants of matrices involving these moments and $2m+1$-step relations between them, thus leading to $2m+1$-band matrices L. Given a Darboux transformations on L, which effect does it have on the m-periodic sequence of weights and on the associated polynomials ? These questions will receive a precise answer in this paper. The methods are based on introducing time parameters in the weights, making the band matrix L evolve according to the so-called discrete KP hierarchy. Darboux transformations on that L translate into vertex operators acting on the τ-function.

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Classical situation: a weight and tridiagonal matrices. A single weight $\rho(z),\ z\in\mathbf{R}$, naturally leads to a moment matrix,

$$m_n=(\mu_{ij})_{0\le i,j\le n-1}=\Big(\langle z^i,z^j\rho(z)\rangle\Big)_{0\le i,j\le n-1}=\Big(\langle z^i,\rho_j(z)\rangle\Big)_{0\le i,j\le n-1},$$

where $\langle f,g\rangle=\int_{\mathbf{R}}fg\,dz$ and where $\rho_j(z):=z^j\rho(z)$. In turn, the moments lead to a sequence of monic orthogonal polynomials

$$p_n(z)\quad=\quad\frac{1}{\det m_n}\det\left(\begin{array}{ccc|c}\mu_{00}&\cdots&\mu_{0,n-1}&1\\ \vdots&&\vdots&\vdots\\ \mu_{n-1,0}&\cdots&\mu_{n-1,n-1}&z^{n-1}\\ \hline \mu_{n0}&\cdots&\mu_{n,n-1}&z^n\end{array}\right),$$

thus satisfying

$$\int_{\mathbf{R}}p_k(z)p_\ell(z)\rho(z)dz=\delta_{k\ell}h_k.$$

Then, as is classically well known, the vector $p(z)=(p_0(z),p_1(z),p_2(z),...)$ of polynomials leads to tridiagonal matrices L, defined by $zp(z)=Lp(z)$.

Periodic sequences of weights and $2m+1$-band matrices. Instead of the classical situation, where $\rho_j(z)=z^j\rho(z)$, we consider an "*m-periodic*" sequence of weights $\rho(z):=(\rho_j(z))_{j\ge 0}$ on $\mathbf{R}$; i.e., satisfying

$$z^m\rho_j(z)=\rho_{j+m}(z); \tag{0.1}$$

in other words,

$$\rho=\Big(\rho_0,\rho_1,...,\rho_{m-1},z^m\rho_0,...,z^m\rho_{m-1},z^{2m}\rho_0,...,z^{2m}\rho_{m-1},...\Big). \tag{0.2}$$

This leads naturally to a $2m+1$-band matrix! Indeed, to this sequence and the inner-product $\langle f,g\rangle=\int_{\mathbf{R}}fg\,dz$, we associate, by analogy, the semi-infinite "*moment matrix*" $m_\infty(\rho)$, where

$$m_n(\rho):=(\mu_{ij}(\rho))_{0\le i,j\le n-1}:=\Big(\Big\langle z^i,\rho_j(z)\Big\rangle\Big)_{0\le i,j\le n-1}, \tag{0.3}$$

the determinant

$$D_n(\rho) := \det m_n(\rho),$$

and the infinite sequence of monic polynomials, where $\mu_{ij} = \mu_{ij}(\rho)$,

$$\begin{aligned} p_n(z) &= \frac{1}{D_n(\rho)} \det \left(\begin{array}{ccc|c} \mu_{00} & \cdots & \mu_{0,n-1} & 1 \\ \vdots & & \vdots & \vdots \\ \mu_{n-1,0} & \cdots & \mu_{n-1,n-1} & z^{n-1} \\ \hline \mu_{n0} & \cdots & \mu_{n,n-1} & z^n \end{array} \right) \\ &= \frac{1}{D_n(\rho)} \det(z\mu_{ij} - \mu_{i+1,j})_{0\le i,j\le n-1}. \end{aligned} \tag{0.4}$$

The second formula for $p_n(z)$ will be discussed in Lemma 2.2. Throughout the paper, the $D_n(\rho)$'s are assumed to be non-zero. Then the sequence $p_n(z)$ gives rise to a semi-infinite matrix L, defined by

$$z^m p(z) = Lp(z), \tag{0.5}$$

where L is a $2m+1$-*band matrix*[1]; this was established by us in [6] and a sketch of the proof will be given in Proposition 2.3. Moreover, Grünbaum and Haine [15] had produced a sequence of "5-step polynomials", satisfying a fourth order differential equation and related to the classical Krall orthonormal polynomials. As we shall see, these polynomials are very special cases of our theory. We conjecture that all sequences of polynomials satisfying $2m+1$-step relations of the precise form (0.5) are given by *generalized periodic sequences of weights*, a slight generalization of (0.1), and limiting cases thereof. See definition 3.2.

In Theorems 0.1 and 0.2, we shall be conjugating with the following matrices:

$$\beta\Lambda^0 + \Lambda = \begin{pmatrix} \beta_0 & 1 & 0 & 0 & \\ 0 & \beta_1 & 1 & 0 & \\ 0 & 0 & \beta_2 & 1 & \\ 0 & 0 & 0 & \beta_3 & \\ & & & & \ddots \end{pmatrix} \quad \text{and} \quad \Lambda^\top\beta + I = \begin{pmatrix} 1 & 0 & 0 & 0 & \\ \beta_0 & 1 & 0 & 0 & \\ 0 & \beta_1 & 1 & 0 & \\ 0 & 0 & \beta_2 & 1 & \\ & & & & \ddots \end{pmatrix},$$

where Λ is the semi-infinite shift matrix $\Lambda := (\delta_{i,j-1})_{i,j\ge 0}$, i.e., $(\Lambda v)_n = v_{n+1}$. Note, in the semi-infinite case, $\Lambda\Lambda^\top = I \neq \Lambda^\top\Lambda$.

Theorem 0.1 (LU-Darboux transforms) *The Lower-Upper Darboux transform*

$$L - \lambda^m I \longmapsto \tilde{L} - \lambda^m I := (\beta\Lambda^0 + \Lambda)(L - \lambda^m I)(\beta\Lambda^0 + \Lambda)^{-1} \tag{0.6}$$

maps L into a new $2m+1$-band matrix $\tilde{L}$, provided

$$\beta_n = -\frac{\Phi_{n+1}(\lambda)}{\Phi_n(\lambda)} \quad \textit{with arbitrary} \quad \Phi(\lambda) = (\Phi_n(\lambda))_{n\ge 0} \in (L - \lambda^m I)^{-1}(0,0,\ldots).$$

[1] zero everywhere, except for m consecutive subdiagonals on either side of the main diagonal.

The null-space $(L-\lambda^m I)^{-1}(0,0,...)$ *is* m*-dimensional with basis vectors given by*

$$\Phi^{(k)}(\lambda)=\left(\frac{D_n\left(\tilde\rho^{(k)}\right)}{D_n(\rho)}\right)_{n\geq 0}, \quad \text{for } 1\leq k\leq m,$$

where

$$\tilde\rho^{(k)}(z):=(\omega^k\lambda-z)\rho(z)=\left((\omega^k\lambda-z)\rho_0(z),\ (\omega^k\lambda-z)\rho_1(z),\dots\right). \tag{0.7}$$

The LU-Darboux transformation $L-\lambda^m I\longmapsto \tilde L-\lambda^m I$ *associated with each*

$$\beta_n=-\frac{\Phi^{(k)}_{n+1}(\lambda)}{\Phi^{(k)}_n(\lambda)}, \quad \text{for fixed } 1\leq k\leq m,$$

induces a map on m*-periodic weights*

$$\rho(z)\longmapsto\tilde\rho^{(k)}(z), \tag{0.8}$$

with $\tilde\rho^{(k)}$ *leading to the* $2m+1$*-band matrix* $\tilde L$*.*

Remark: Section 5 (Theorem 5.1) contains the proof of a more general statement, involving linear combinations of $\Phi^{(k)}(\lambda)$.

Theorem 0.2 (UL-Darboux transforms). *The Upper-Lower Darboux transform*

$$L-\lambda^m I\longmapsto\tilde L-\lambda^m I:=(\Lambda^\top\beta+I)(L-\lambda^m I)(\Lambda^\top\beta+I)^{-1}, \tag{0.9}$$

maps L *into a new* $2m+1$*-band matrix* $\tilde L$*, provided*[2]

$$\beta_n=-\frac{\Phi_{n+1}(\lambda)}{\Phi_n(\lambda)} \quad \text{with} \quad \Phi(\lambda)=(\Phi_n(\lambda))_{n\geq 0}\in(L-\lambda^m I)^{-1}\operatorname{span}(e_1,e_2,\dots,e_m).$$

The(quasi)-null vectors $\Phi(\lambda)$ *of* $L-\lambda^m I$ *depends projectively on* $2m-1$ *free parameters* $a_0,...,a_{m-1},b_0,...,b_{m-1}$ [3] *and are given by*

$$\Phi(\lambda)=\left((-1)^{n-1}\frac{D_{n+1}(\tilde\rho)}{D_n(\rho)}\right)_{n\geq 0}, \tag{0.10}$$

where

$$\tilde\rho=\left(\tilde\rho_0,\tilde\rho_1,...,\tilde\rho_{m-1},z^m\tilde\rho_0,...,z^m\tilde\rho_{m-1},z^{2m}\tilde\rho_0,...,z^{2m}\tilde\rho_{m-1},...\right), \tag{0.11}$$

[2] $e_i:=(0,\dots,\underbrace{1}_{i},0,\dots)\in\mathbf{R}^\infty$.

[3] The UL-Darboux transform depends on m additional free parameters, compared to the LU transform .

with[4]

$$\begin{aligned}\tilde{\rho}_0(z) : &= \sum_{k=0}^{m-1}\left(a_k\delta(z-\omega^k\lambda)+b_k\frac{\rho_k(z)}{z^m-\lambda^m}\right), \quad \textit{with} \quad b_{m-1}\neq 0,\\ \tilde{\rho}_k(z) : &= \rho_{k-1}(z), \qquad \textit{for} \quad 1\leq k\leq m-1.\end{aligned} \tag{0.12}$$

The UL-Darboux transform $L-\lambda^m I \longmapsto \tilde{L}-\lambda^m I$ induces a map on m-periodic sequence of weights

$$\rho \longmapsto \tilde{\rho},$$

with $\tilde{\rho}$ leading to the $2m+1$-band matrix $\tilde{L}$.

Corollary 0.3 *An appropriate choice of a_k, and appropriate limits $b_k \mapsto \infty$ and $\lambda \mapsto 0$ yield the following special Darboux transformation on the m-periodic weights*

$$\rho = (\rho_0, \rho_1, \ldots) \mapsto \tilde{\rho} = (\tilde{\rho}_0, \tilde{\rho}_1, \tilde{\rho}_2, \ldots),$$

with new weights

$$\begin{aligned}\tilde{\rho}_0(z) : &= \sum_{k=0}^{m-1}\left(c_k\left(\frac{d}{dz}\right)^k\delta(z)+d_k\frac{\rho_k(z)}{z^m}\right), \ \textit{with } d_{m-1}\neq 0,\\ \tilde{\rho}_k(z) : &= \rho_{k-1}(z) \qquad \textit{for} \quad 1\leq k\leq m-1.\end{aligned}$$

Weights with δ-functions have been studied mainly by Krall and Scheffer [21] and Koornwinder [18], at least for the standard orthogonal polynomials. For recent expositions on the subject, see, for instance, Andrews and Askey [10]. Recently, they have been studied by Grünbaum-Haine [15] and Grünbaum-Haine-Horozov [16].

An integrable flow with initial m_∞ . We have introduced the method of inserting the time in the context of random matrices [8, 9, 24], where it has turned out to be very useful. In order to establish the results above, consider, as we did in [3, 4], the initial value problem, depending on two sequences of time parameters $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$:

$$\frac{\partial m_\infty}{\partial x_n} = \Lambda^n m_\infty, \quad \frac{\partial m_\infty}{\partial y_n} = -m_\infty\Lambda^{\top n}, \text{ with initial } m_\infty(0,0) = (\langle z^i, \rho_j(z)\rangle)_{0\leq i,j<\infty}, \tag{0.13}$$

where Λ is the customary (semi-infinite) shift matrix. As we shall establish in section 2, imposing the condition

$$\Lambda^m m_\infty = m_\infty \Lambda^{\top m} \tag{0.14}$$

[4]the delta-function is defined in the standard way $\int f(z)\delta(\lambda - z)dz = f(\lambda)$.

on moment matrices m_∞ leads to $2m+1$-band matrices. This in turn, suggests the following useful reduction: given the times $x, y \in \mathbf{C}^\infty$, we define new *times* $\bar{x}, \bar{y}, \bar{t} \in \mathbf{C}^\infty$,

$$\begin{aligned} \bar{x} &= (x_1, ..., x_{m-1}, 0, x_{m+1}, ..., x_{2m-1}, 0, x_{2m+1}, ...) \\ \bar{y} &= (y_1, ..., y_{m-1}, 0, y_{m+1}, ..., y_{2m-1}, 0, y_{2m+1}, ...) \\ \bar{t} &= (0, ..., 0, t_m, 0, ..., 0, t_{2m}, 0, ..., 0, t_{3m}, 0, ...), \end{aligned}$$

with

$$t_{km} := x_{km} - y_{km} \quad \text{for} \quad k = 1, 2, ... \ . \tag{0.15}$$

The point is that, letting m_∞ evolve according to the variables $\bar{x}, \bar{y}, \bar{t}$, will conserve the $2m+1$-band form of L. The solution to the initial value problem (0.13) is given by the same moment matrix m_∞, as in (0.13),

$$m_\infty\left(\rho(z; \bar{x}, \bar{y}, \bar{t})\right) = \left(\left\langle z^i, \rho_j(z; \bar{x}, \bar{y}, \bar{t})\right\rangle\right)_{0 \le i,j < \infty}, \tag{0.16}$$

but for weights, now depending on times $\bar{x}, \bar{y}, \bar{t}$, defined as [5]

$$\rho_j(z; \bar{x}, \bar{y}, \bar{t}) = e^{\sum_1^\infty \bar{x}_r z^r} e^{\sum_1^\infty \bar{t}_{\ell m} z^{\ell m}} \sum_{\ell=0}^\infty \mathbf{s}_\ell(-\bar{y}) \rho_{j+\ell}(z), \tag{0.17}$$

in terms of the initial condition $\rho(z)$. The moments (0.16) give rise to the polynomials $p_n(z; \bar{x}, \bar{y}, \bar{t})$, as in (0.4), which, in turn, give rise to $2m+1$-band matrices L, via $z^m p = Lp$. Then L satisfies the following equations[6] in the time parameters $(\bar{x}, \bar{y}, \bar{t})$,

$$\frac{\partial L}{\partial x_i} = [(\overline{L^{i/m}})_+, L], \quad \frac{\partial L}{\partial y_i} = [(\underline{L^{i/m}})_-, L], \text{ for } i = 1, 2, ..., m \nmid i$$

$$\frac{\partial L}{\partial t_{im}} = [(L^i)_+, L], \quad i = 1, 2, ... \, . \tag{0.18}$$

Vertex operators. In order to obtain formulae (0.7) and (0.12) for the weights, we consider two vertex operators, naturally associated with the integrable system (0.13) for $2m+1$-band matrices[7]

$$\begin{aligned} \mathbf{X}_1(\lambda) &:= \chi(\lambda) e^{\sum_1^\infty \bar{t}_{mi} \lambda^{mi}} e^{-\sum_1^\infty \frac{\lambda^{-mi}}{mi} \frac{\partial}{\partial t_{mi}}} e^{\sum_1^\infty \bar{x}_i \lambda^i} e^{-\sum_1^\infty \frac{\lambda^{-i}}{i} \frac{\partial}{\partial \bar{x}_i}} \\ \mathbf{X}_2(\lambda) &:= \chi(\lambda^{-1}) e^{-\sum_1^\infty \bar{t}_{mi} \lambda^{mi}} e^{\sum_1^\infty \frac{\lambda^{-mi}}{mi} \frac{\partial}{\partial \bar{t}_{mi}}} e^{\sum_1^\infty \bar{y}_i \lambda^i} e^{-\sum_1^\infty \frac{\lambda^{-i}}{i} \frac{\partial}{\partial \bar{y}_i}} \Lambda. \end{aligned} \tag{0.19}$$

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[5] The $\mathbf{s}_\ell$'s denote the elementary Schur polynomials $e^{\sum_1^\infty t_i z^i} = \sum_0^\infty \mathbf{s}_n(t) z^n$.

[6] Note $\overline{L^{1/m}}$ and $\underline{L^{1/m}}$ are the *right* m^{th} roots and *left* m^{th} roots, so that:

$$\begin{aligned} \overline{L^{i/m}} &= (\overline{L^{1/m}})^i \ \text{ where } \ \overline{L^{1/m}} = \Lambda + \sum_{k \le 0} b_k \Lambda^k \\ \underline{L^{i/m}} &= (\underline{L^{1/m}})^i \ \text{ where } \ \underline{L^{1/m}} = c_{-1}\Lambda^{-1} + \sum_{k \ge 0} c_k \Lambda^k. \end{aligned}$$

[7] $\chi(\lambda) = \text{diag}\,(\lambda^0, \lambda, \lambda^2, ...)$.

The vertex operators (0.19) act on vectors of functions $\tau(\bar{x},\bar{y},\bar{t}) = (\tau_n(\bar{x},\bar{y},\bar{t}))_{n\geq 0}$. In [5], we showed general linear combinations of them are the precise implementation of the Darboux transform (0.6) and (0.9) at the level of τ-functions; see theorems 4.1 and 4.2. Then in the end, we set $(\bar{x},\bar{y},\bar{t}) = (0,0,0)$, which yield formulae (0.7) and (0.12) for the new weights.

It is well-known that the vertex operators generate Virasoro-like symmetries at the level of the τ-functions, which translate into symmetries at the level of the "wave"-functions for band matrices. For the study of such symmetries, see Dickey [13, 14] and [7]. For an extensive exposition on Darboux transforms, see the book by Matveev and Salle [22].

Examples: 1. Tridiagonal matrices: A single weight leads to a moment matrix m_∞ with $\Lambda m_\infty = m_\infty \Lambda$ and a tridiagonal matrix L; formulae (0.15) reduce to one set of times $t := \bar{t} = (t_1, t_2, \ldots)$. Equations (0.18) become the standard Toda lattice, with τ-functions

$$\tau_n(t) = \det m_n \left(\rho(z) e^{\sum_1^\infty t_i z^i}\right). \tag{0.20}$$

The standard Toda lattice vertex operator, introduced by us in [2] and obtained from (0.19),

$$\mathbf{X}(t,\lambda) = \Lambda^{-1}\chi(\lambda^2) e^{\sum_1^\infty t_i\lambda^i} e^{-2\sum_1^\infty \frac{\lambda^{-i}}{i}\frac{\partial}{\partial t_i}} \tag{0.21}$$

has the surprising property that, given a Toda τ-vector $\tau(t) = (\tau_0, \tau_1, \ldots)$, the vector[8]

$$\tau(t) + c\mathbf{X}(t,\lambda)\tau(t) = \left(\tau_n(t) + c\lambda^{2n-2} e^{\sum_1^\infty t_i\lambda^i}\tau_{n-1}(t - 2[\lambda^{-1}])\right)_{n\geq 0} \tag{0.22}$$

is again a Toda τ-vector. This precise operation can be implemented by a UL-Darboux transform, followed by a LU-Darboux transform and a limit. Note the UL-Darboux transform (resp. LU-Darboux transform) amounts, for a tridiagonal matrix, to a factorization of $L - \lambda I$ into an upper- times a lower-triangular matrix (resp. lower- times an upper-triangular matrix), and to multiplying the factors in the opposite order. The vertex operator above translates into adding a delta-function to the original weight. This establishes a dictionary between several points of view (explained in section 6) :

$$L - \lambda = L_+ L_- \longmapsto L' - \lambda := L_- L_+ \longmapsto L' - \mu = L'_- L'_+ \longmapsto L'' - \mu := L'_+ L'_-,$$

$$\updownarrow$$

$$\rho(z) \longmapsto \rho(z) + c\delta(\lambda - z)$$

$$\updownarrow$$

$$\tau + c\mathbf{X}\tau \tag{0.23}$$

[8] For $\alpha \in \mathbf{C}$, define $[\alpha] = (\frac{\alpha}{1}, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \ldots) \in \mathbf{C}^\infty$.

2. "Classical" polynomials, satisfying $2m+1$-step relations: Given moments $\mu_i := \langle z^i, \rho_0(z)\rangle$, associated with a single weight ρ_0 for standard orthogonal polynomials, satisfying for fixed integer $m \geq 1$,

$$\int_{\mathbf{R}} \left| z^j \rho_0(z) \right| dz < \infty, \quad j \geq -m+1,$$

we define in section 7 new monic polynomials $\tilde{p}_n^{(1)}(z)$, defined by a new moment matrix $\tilde{m}_\infty$, which coincides with the old moment matrix $m_\infty = (\mu_{i+j})_{i,j\geq 0}$ associated with the standard orthogonal polynomials, except for the first column. The $\tilde{p}_n^{(1)}(z)$, defined by

$(\det \tilde{m}_n)\ \tilde{p}_n^{(1)}(z) =$

$$\det \begin{pmatrix} \sum_{k=0}^{m-1} \mu_{-k} d_{m-k-1} + c_0 & \mu_1 & \mu_2 & \ldots & 1 \\ \sum_{k=0}^{m-1} \mu_{1-k} d_{m-k-1} - c_1 & \mu_2 & \mu_3 & \ldots & z \\ \sum_{k=0}^{m-1} \mu_{2-k} d_{m-k-1} + 2! c_2 & \mu_3 & \mu_4 & \ldots & z^2 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \sum_{k=0}^{m-1} \mu_{m-k-1} d_{m-k-1} + (-1)^{m-1}(m-1)! c_{m-1} & \mu_m & \mu_{m+1} & \ldots & z^{m-1} \\ \sum_{k=0}^{m-1} \mu_{m-k} d_{m-k-1} & \mu_{m+1} & \mu_{m+2} & \ldots & z^m \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \sum_{k=0}^{m-1} \mu_{n-k} d_{m-k-1} & \mu_{n+1} & \mu_{n+2} & \ldots & z^n \end{pmatrix},$$

satisfy $2m+1$-step relations, i.e.,

$$z^m p^{(1)}(z) = L p^{(1)}(z), \quad \text{with a } 2m+1\text{- band matrix } L.$$

It remains an interesting open question to find out whether such polynomials satisfy differential equations; on such matters, see section 7.

1 Borel decomposition and the 2-Toda lattice

In [3, 4], we considered the following differential equations for the bi-infinite or semi-infinite matrix m_∞

$$\frac{\partial m_\infty}{\partial x_n} = \Lambda^n m_\infty, \quad \frac{\partial m_\infty}{\partial y_n} = -m_\infty \Lambda^{\top n}, \quad n = 1, 2, \ldots, \tag{1.1}$$

where the matrix $\Lambda = (\delta_{i,j-1})_{i,j\in\mathbf{Z}}$ is the shift matrix; then (1.1) has the following solutions:

$$m_\infty(x,y) = e^{\sum_1^\infty x_n\Lambda^n} m_\infty(0,0) e^{-\sum_1^\infty y_n\Lambda^{\top n}} \tag{1.2}$$

in terms of some initial condition $m_\infty(0,0)$. In this general setup, the matrix m_∞ is a general matrix, thus not necessarily generated by weights ρ.

Consider the Borel decomposition $m_\infty = S_1^{-1}S_2$, for

$$\begin{aligned} S_1 \in G_- &= \{\text{lower-triangular invertible matrices, with 1's on the diagonal}\} \\ S_2 \in G_+ &= \{\text{upper-triangular invertible matrices}\}, \end{aligned}$$

with corresponding Lie algebras g_-, g_+; then setting $\mathcal{L}_1 := S_1\Lambda S^{-1}$,

$$\begin{aligned} S_1\frac{\partial m_\infty}{\partial x_n}S_2^{-1} &= S_1\frac{\partial S_1^{-1}S_2}{\partial x_n}S_2^{-1} = -\frac{\partial S_1}{\partial x_n}S_1^{-1} + \frac{\partial S_2}{\partial x_n}S_2^{-1} \in g_- + g_+ \\ &= S_1\Lambda^n m_\infty S_2^{-1} = S_1\Lambda^n S_1^{-1} = \mathcal{L}_1^n = (\mathcal{L}_1^n)_- + (\mathcal{L}_1^n)_+ \in g_- + g_+; \end{aligned}$$

the uniqueness of the decomposition $g_- + g_+$ leads to

$$-\frac{\partial S_1}{\partial x_n}S_1^{-1} = (\mathcal{L}_1^n)_-, \quad \frac{\partial S_2}{\partial x_n}S_2^{-1} = (\mathcal{L}_1^n)_+.$$

Similarly setting $\mathcal{L}_2 = S_2\Lambda^\top S_2^{-1}$, we find

$$-\frac{\partial S_1}{\partial y_n}S_1^{-1} = -(\mathcal{L}_2^n)_-, \quad \frac{\partial S_2}{\partial y_n}S_2^{-1} = -(\mathcal{L}_2^n)_+.$$

This leads to the 2-Toda equations for S_1, S_2 and $\mathcal{L}_1, \mathcal{L}_2$:

$$\frac{\partial S_{1,2}}{\partial x_n} = \mp(\mathcal{L}_1^n)_\mp S_{1,2}, \quad \frac{\partial S_{1,2}}{\partial y_n} = \pm(\mathcal{L}_2^n)_\mp S_{1,2} \tag{1.3}$$

$$\frac{\partial \mathcal{L}_i}{\partial x_n} = [(\mathcal{L}_1^n)_+, \mathcal{L}_i], \quad \frac{\partial \mathcal{L}_i}{\partial y_n} = [(\mathcal{L}_2^n)_-, \mathcal{L}_i], \quad i = 1,2,... \tag{1.4}$$

By 2-Toda theory [4] the problem is solved in terms of a sequence of tau-functions

$$\tau_n(x,y) = \det m_n(x,y), \tag{1.5}$$

with $m_n(x,y)$ defined below:
<u>bi-infinite case</u> ($n \in \mathbf{Z}$):

$$m_n(x,y) := (\mu_{ij}(x,y))_{-\infty<i,j\le n-1},$$

<u>semi-infinite case</u> ($n \ge 0$):

$$m_n(x,y) := (\mu_{ij}(x,y))_{0\le i,j\le n-1}, \quad \text{with } \tau_0 = 1. \tag{1.6}$$

The two pairs of wave functions $\Psi=(\Psi_1,\Psi_2)$ and $\Psi^*=(\Psi_1^*,\Psi_2^*)$ defined by [9]

$$\Psi_1(z;x,y)=e^{\sum_1^\infty x_iz^i}S_1\chi(z),\quad \Psi_1^*(z;x,y)=e^{-\sum_1^\infty x_iz^i}\left(S_1^\top\right)^{-1}\chi(z^{-1})$$

$$\Psi_2(z;x,y)=e^{\sum_1^\infty y_iz^{-i}}S_2\chi(z),\quad \Psi_2^*(z;x,y)=e^{-\sum_1^\infty y_iz^{-i}}\left(S_2^\top\right)^{-1}\chi(z^{-1}) \tag{1.7}$$

satisfy

$$\mathcal{L}_1\Psi_1=z\Psi_1,\ \mathcal{L}_2\Psi_2=z^{-1}\Psi_2,\quad \mathcal{L}_1^\top\Psi_1^*=z\Psi_1^*,\ \mathcal{L}_2^\top\Psi_2^*=z^{-1}\Psi_2^*,$$

and

$$\left\{\begin{array}{l}\frac{\partial}{\partial x_n}\Psi_i=(\mathcal{L}_1^n)_+\Psi_i\\ \frac{\partial}{\partial y_n}\Psi_i=(\mathcal{L}_2^n)_-\Psi_i\end{array}\right.\qquad \left\{\begin{array}{l}\frac{\partial}{\partial x_n}\Psi_i^*=-((\mathcal{L}_1^n)_+)^\top\Psi_i^*\\ \frac{\partial}{\partial y_n}\Psi_i^*=-((\mathcal{L}_2^n)_-)^\top\Psi_i^*.\end{array}\right. \tag{1.8}$$

In [23], with a slight notational modification [8], the wave functions are shown to have the following τ-function representation:

$$\begin{aligned}
\Psi_1(z;x,y)&=\left(\frac{\tau_n(x-[z^{-1}],y)}{\tau_n(x,y)}e^{\sum_1^\infty x_iz^i}z^n\right)_{n\in\mathbf{Z}}\\
\Psi_2(z;x,y)&=\left(\frac{\tau_{n+1}(x,y-[z])}{\tau_n(x,y)}e^{\sum_1^\infty y_iz^{-i}}z^n\right)_{n\in\mathbf{Z}}\\
\Psi_1^*(z;x,y)&=\left(\frac{\tau_{n+1}(x+[z^{-1}],y)}{\tau_{n+1}(x,y)}e^{-\sum_1^\infty x_iz^i}z^{-n}\right)_{n\in\mathbf{Z}}\\
\Psi_2^*(z;x,y)&=\left(\frac{\tau_n(x,y+[z])}{\tau_{n+1}(x,y)}e^{-\sum_1^\infty y_iz^{-i}}z^{-n}\right)_{n\in\mathbf{Z}},
\end{aligned} \tag{1.9}$$

with the following bilinear identities satisfied for the wave and adjoint wave functions Ψ and Ψ^*, for all $m,n\in\mathbf{Z}$ (bi-infinite) and $m,n\geq 0$ (semi-infinite) and $x,y,x',y'\in\mathbf{C}^\infty$:

$$\oint_{z=\infty}\Psi_{1n}(z;x,y)\Psi_{1m}^*(z;x',y')\frac{dz}{2\pi iz}=\oint_{z=0}\Psi_{2n}(z;x,y)\Psi_{2m}^*(z;x',y')\frac{dz}{2\pi iz}. \tag{1.10}$$

The τ-functions[10] satisfy the following bilinear identities:

$$\begin{aligned}
&\oint_{z=\infty}\tau_n(x-[z^{-1}],y)\tau_{m+1}(x'+[z^{-1}],y')e^{\sum_1^\infty(x_i-x_i')z^i}z^{n-m-1}dz\\
&=\oint_{z=0}\tau_{n+1}(x,y-[z])\tau_m(x',y'+[z])e^{\sum_1^\infty(y_i-y_i')z^{-i}}z^{n-m-1}dz;
\end{aligned} \tag{1.11}$$

[9]In this section,

$$\begin{aligned}\chi(z)&=\text{diag }(\ldots,z^{-1},z^0,z^1,\ldots)\text{ in the bi-infinite case}\\ &=\text{diag }(z^0,z^1,\ldots)\ \text{ in the semi-infinite case.}\end{aligned}$$

[10]The first contour runs clockwise about a small neighborhood of $z=\infty$, while the second runs counter-clockwise about $z=0$.

they characterize the 2-Toda lattice τ-functions. Note (1.7) and (1.9) yield

$$(S_2)_0 = \mathrm{diag}(..., \frac{\tau_{n+1}(x,y)}{\tau_n(x,y)}, ...) := h(x,y). \tag{1.12}$$

In [23], the facts above are shown for the bi-infinite case; they can be carefully specialized to the semi-infinite case, upon setting $\tau_{-i} = 0$ for $i = 1, 2, ...$

Consider the usual inner-product $\langle , \rangle$ and an infinite sequence of weights $\rho(z) = (\rho_0(z), \rho_1(z), ...)$. The moment matrix $m_\infty = m_\infty(\rho(z))$ will now depend on $\rho(z)$. The following proposition will play an important role in this paper.

Proposition 1.1 *The solution to the equations*

$$\frac{\partial m_\infty}{\partial x_n} = \Lambda^n m_\infty, \quad \frac{\partial m_\infty}{\partial y_n} = -m_\infty \Lambda^{\top n}, \quad n = 1, 2, ..., \tag{1.13}$$

with initial condition

$$m_\infty(\rho(z;0,0)) = (\langle z^i, \rho_j(z)\rangle)_{0\le i,j\le\infty},$$

is given by

$$m_\infty = \left(\langle z^i, \rho_j(z;x,y)\rangle\right)_{i,j\ge 0}, \tag{1.14}$$

where the weights $\rho_j(z;x,y)$ evolve as follows[11]

$$\rho_j(z;x,y) = e^{\sum_1^\infty x_i z^i} \sum_{\ell=0}^\infty \mathbf{s}_\ell(-y)\rho_{j+\ell}(z), \tag{1.15}$$

in terms of the initial condition $\rho(z;0,0) = (\rho_0(z), \rho_1(z), ...)$.

<u>*Proof*</u>: Indeed, one checks, from (1.15),

$$\frac{\partial \rho_j}{\partial x_k} = z^k \rho_j(z;x,y) \quad \frac{\partial \rho_j}{\partial y_k} = -e^{\sum_1^\infty x_i z^i} \sum_{\ell=k}^\infty \mathbf{s}_{\ell-k}(-y)\rho_{j+\ell}(z) = -\rho_{j+k}(z;x,y),$$

from which it follows that

$$\frac{\partial}{\partial x_k}\mu_{ij}(\rho(z;x,y)) = \frac{\partial}{\partial x_k}\langle z^i, \rho_j(z;x,y)\rangle = \langle z^{i+k}, \rho_j(z;x,y)\rangle = \mu_{i+k,j}(\rho(z;x,y)),$$

$$\frac{\partial}{\partial y_k}\mu_{ij}(\rho(z;x,y)) = \frac{\partial}{\partial y_k}\langle z^i, \rho_j(z:x,y)\rangle = -\left\langle z^i, \rho_{j+k}(z;x,y)\right\rangle = -\mu_{i,j+k}(\rho(z;x,y)),$$

which is equivalent to (1.13). Here is an alternative way of checking this fact: since, from (1.14),

$$(\Lambda^k m_\infty(\rho(z;x,y)))_{ij} = \langle z^{i+k}, \rho_j(z;x,y)\rangle \quad \text{and} \quad (m_\infty(\rho(z;x,y))\Lambda^{\top k})_{ij} = \langle z^i, \rho_{j+k}(z;x,y)\rangle,$$

one checks

[11] The elementary Schur polynomials are defined in footnote 4; also $\frac{\partial \mathbf{s}_i}{\partial x_k} = \mathbf{s}_{i-k}$.

$$
\begin{aligned}
&e^{\sum_1^\infty x_n\Lambda^n}\langle z^i,\rho_j(z;0,0)\rangle_{0\le i,j\le\infty}e^{-\sum_1^\infty y_n\Lambda^{\top n}}\\
&\qquad = \sum_{k=0}^{\infty}\mathbf{s}_k(x)\Lambda^k\left\langle z^i,\rho_j(z;0,0)\right\rangle_{0\le i,j<\infty}\sum_{\ell=0}^{\infty}\mathbf{s}_\ell(-y)\Lambda^{\top\ell}\\
&\qquad = \sum_{k,\ell=0}^{\infty}\mathbf{s}_k(x)\ \left\langle z^{i+k},\rho_{j+\ell}(z;0,0)\right\rangle_{0\le i,j<\infty}\ \mathbf{s}_\ell(-y)\\
&\qquad = \left\langle e^{\sum_1^\infty x_kz^k}z^i,\sum_{\ell=0}^{\infty}\mathbf{s}_\ell(-y)\rho_{j+\ell}(z;0,0)\right\rangle_{0\le i,j<\infty}\\
&\qquad = \left\langle z^i,\rho_j(z;x,y)\right\rangle_{0\le i,j<\infty}. \qquad (1.16)
\end{aligned}
$$

■

2 Reductions of the 2-Toda Lattice

Reduction from 2-Toda to $2m+1$-band matrices :

For convenience, we define new vectors $\bar{x},\bar{y},\bar{t}\in\mathbf{C}^\infty$, based on the vectors $x,y\in\mathbf{C}^\infty$:

$$
\begin{aligned}
\bar{x} &= (x_1,...,x_{m-1},0,x_{m+1},...,x_{2m-1},0,x_{2m+1},...)\\
\bar{y} &= (y_1,...,y_{m-1},0,y_{m+1},...,y_{2m-1},0,y_{2m+1},...)\\
\bar{t} &= (0,...,0,t_m,0,...,0,t_{2m},0,...,0,t_{3m},0,...),
\end{aligned}
$$

with

$$t_{km}:=x_{km}-y_{km}\text{ for }k=1,2,...\ . \qquad (2.1)$$

Notice in this subsection, $\mathcal{L}_1$ and $\mathcal{L}_2$ are bi-infinite. In the next subsection, we shall specialize this to the semi-infinite case.

Recall from section 1,

$$m_\infty=S_1^{-1}S_2,\ \mathcal{L}_1=S_1\Lambda S_1^{-1},\ \mathcal{L}_2=S_2\Lambda^\top S_2^{-1}\text{ and }\tau_n=\det m_n.$$

Proposition 2.1 *Whenever $\tau_n(x,y)\neq 0$ for all $n\in\mathbf{Z}$, the following three statements are equivalent:*
(i) $\Lambda^m m_\infty=m_\infty\Lambda^{\top m}$
(ii) $\mathcal{L}_1^m=\mathcal{L}_2^m$*, in which case* $\mathcal{L}_1^m$ *is a $2m+1$-band matrix.*
(iii) $\mathcal{L}_1$*,* $\mathcal{L}_2$*,* m_∞ *and* τ_n *are functions of* $\bar{x}$*,*$\bar{y}$ *and* $\bar{t}$ *only.*
Also (i) or (ii) are invariant manifolds of the vector fields $\frac{\partial m_\infty}{\partial x_n}=\Lambda^n m_\infty$, $\frac{\partial m_\infty}{\partial y_n}=-m_\infty\Lambda^{\top n}$, $n=1,2,...,$.

Proof: Indeed, by the invertibility of S_1 and S_2 under the proviso above, and remembering the splitting $m_\infty = S_1^{-1}S_2$, we have that (i) holds if and only if

$$\mathcal{L}_1^m = S_1\Lambda^m S_1^{-1} = S_1\Lambda^m m_\infty S_2^{-1} = S_1 m_\infty \Lambda^{\top m} S_2^{-1} = S_2\Lambda^{\top m}S_2^{-1} = \mathcal{L}_2^m. \tag{2.2}$$

Also note that (i) is equivalent to

$$0 = \Lambda^{km}m_\infty - m_\infty\Lambda^{\top km} = \left(\frac{\partial}{\partial x_{km}} + \frac{\partial}{\partial y_{kn}}\right) m_\infty, \quad ,k = 1,2,\dots .$$

This is also tantamount to statement (iii), because the invariance of m_∞ under $\partial/\partial x_{km} + \partial/\partial y_{km}$ implies the invariance of $\mathcal{L}_1$, $\mathcal{L}_2$ and τ_n. From the solution (1.2), if (i) holds at $(x,y) = (0,0)$, it holds for all (x,y), and thus, by (2.2), if (ii) holds at $(0,0)$, it also holds for all (x,y). ■

From Proposition 2.1, it follows that the Toda vector fields respect the band structure of $L := \mathcal{L}_1^m = \mathcal{L}_2^m$, i.e., it is an invariant manifold of the flow. Therefore the Toda theory can be recast purely in terms of the $2m+1$-*band matrix* of the form

$$\begin{aligned} L &= \sum_{-m\le i\le m} a_i\Lambda^i \\ &= \left(\begin{array}{ccc|cccc} \ddots & & \ddots & \ddots & & \ddots & O \\ a_{-m+1}(-1) & \dots & a_0(-1) & a_1(-1) & \dots & 1 & \\ \hline a_{-m}(0) & \dots & a_{-1}(0) & a_0(0) & \dots & a_{m-1}(0) & 1 \\ O & \ddots & & \ddots & & \ddots & \ddots \end{array}\right), \end{aligned} \tag{2.3}$$

with a_i being diagonal matrices and $a_m = I$. The vector fields below involve the ith powers $\overline{L^{i/m}} = \mathcal{L}_1^i$ and $\underline{L^{i/m}} = \mathcal{L}_2^i$ of the *right* m^{th} roots $\overline{L^{1/m}} = \mathcal{L}_1$ and *left* m^{th} roots $\underline{L^{1/m}} = \mathcal{L}_2$ respectively; see also footnote 6.

The *m-reduced Toda lattice vector fields* on L are as follows:

$$\frac{\partial L}{\partial x_i} = [(\overline{L^{i/m}})_+, L], \quad \frac{\partial L}{\partial y_i} = [(\underline{L^{i/m}})_-, L], \text{ for } i = 1,2,...,m \nmid i$$

$$\frac{\partial L}{\partial t_{im}} = [(L^i)_+, L], \quad i = 1,2,\dots . \tag{2.4}$$

Then L can be expressed in terms of a string of τ-functions

$$\tau_n := \tau_n(\bar{x},\bar{y},\bar{t}), \tag{2.5}$$

which in the semi-infinite case will take on a very concrete form.

Reduction from bi-infinite to semi-infinite 2-Toda :

In this section we focus on the Borel decomposition of section 1, but specifically for semi-infinite matrices $m_\infty = (\mu_{ij})_{i,j\geq 0}$, where it is unique. Remember the decomposition $m_\infty = S_1^{-1}S_2$, where S_1 is lower-triangular, with 1's on the diagonal and where S_2 is upper-triangular with $h_n = \det(m_{n+1})/\det(m_n)$ on the diagonal, by (1.12). Let h denote such a diagonal matrix. For any matrix m_∞, define $\mathcal{S}(m_\infty) := S_1$ and $h(m_\infty) := h$, as functions of the matrix m_∞. Following [3], we write the Borel decomposition, as follows

$$m_\infty = S_1^{-1}S_2 = (\mathcal{S}(m_\infty))^{-1}h(m_\infty)\left(\mathcal{S}(m_\infty^\top)\right)^{\top -1}. \tag{2.6}$$

It leads naturally to vectors of monic bi-orthogonal polynomials

$$p^{(1)}(z) = \mathcal{S}(m_\infty)\chi(z) = S_1\chi(z) \quad \text{and} \quad p^{(2)}(z) = \mathcal{S}(m_\infty^\top)\chi(z) = h(S_2^\top)^{-1}\chi(z). \tag{2.7}$$

Upon introducing a formal inner-product $\langle\ ,\ \rangle_0$, where $\langle y^i, z^j\rangle_0 = \mu_{ij}$, the polynomials $p^{(1)}(z)$ and $p^{(2)}(z)$ enjoy the following orthogonality property, using (2.6):

$$\left(\langle p_i^{(1)}, p_j^{(2)}\rangle_0\right)_{i,j\geq 0} = S_1 m\left(h(S_2^\top)^{-1}\right)^\top = \mathcal{S}(m_\infty)m_\infty\mathcal{S}(m_\infty^\top)^\top = h. \tag{2.8}$$

Letting the semi-infinite matrix m_∞ evolve according to the differential equations (1.1), namely

$$\frac{\partial m_\infty}{\partial x_n} = \Lambda^n m_\infty, \quad \frac{\partial m_\infty}{\partial y_n} = -m_\infty\Lambda^{\top n}, \quad n = 1, 2, ...,$$

we have shown, in [3], that the wave functions Ψ_1 and Ψ_2^* have the following representation in terms of the bi-orthogonal polynomials constructed from $m_\infty(x,y)$ in (2.7):

$$\Psi_1(z;x,y) = e^{\sum x_k z^k}p^{(1)}(z;x,y) = e^{\sum x_k z^k}S_1\chi(z) \tag{2.9}$$

$$\Psi_2^*(z;x,y) = e^{-\sum y_k z^{-k}}h^{-1}p^{(2)}(z^{-1};x,y) = e^{-\sum y_k z^{-k}}(S_2^{-1})^\top\chi(z^{-1}), \tag{2.10}$$

with the p_n's being expressed in terms of τ-functions τ_n of 2-Toda:

$$p_n^{(1)}(z;x,y) = z^n\frac{\tau_n(x-[z^{-1}],y)}{\tau_n(x,y)}\ , \qquad p_n^{(2)}(z;x,y) = z^n\frac{\tau_n(x,y+[z^{-1}])}{\tau_n(x,y)}. \tag{2.11}$$

and

$$\tau_n(x,y) = \det m_n(x,y) \quad \text{and} \quad h_n = \frac{\tau_{n+1}(x,y)}{\tau_n(x,y)} \tag{2.12}$$

In [3], we have shown the following matrix representation for the bi-orthogonal polynomials, which then leads, using (2.7), to a representation of the lower-triangular matrices $\mathcal{S}(m_\infty)$ and $\mathcal{S}(m_\infty^\top)$:

$$p_n^{(1)}(z;x,y) = \frac{1}{\tau_n(x,y)}\det\left(\begin{array}{ccc|c} \mu_{00} & \cdots & \mu_{0,n-1} & 1 \\ \vdots & & \vdots & \vdots \\ \mu_{n-1,0} & \cdots & \mu_{n-1,n-1} & z^{n-1} \\ \hline \mu_{n,0} & \cdots & \mu_{n,n-1} & z^n \end{array}\right) \tag{2.13}$$

$$p_n^{(2)}(z;x,y) = \frac{1}{\tau_n(x,y)} \det \left(\begin{array}{ccc|c} \mu_{00} & \dots & \mu_{n-1,0} & 1 \\ \vdots & & \vdots & \vdots \\ \mu_{0,n-1} & \dots & \mu_{n-1,n-1} & z^{n-1} \\ \hline \mu_{0,n} & \dots & \mu_{n-1,n} & z^n \end{array} \right) \tag{2.14}$$

Assume now the moments μ_{ij} are given by weights $\rho(z) = (\rho_0(z), \rho_1(z), ...)$; then

$$\tau_n(x,y) = \det \left(\langle z^i, \rho_j(z;x,y) \rangle \right)_{0 \le i,j \le n-1} = D_n(\rho(x,y)),$$

where $\rho_j(z;x,y)$ is given by (1.15), i.e.,

$$\rho_j(z;x,y) = e^{\sum_1^\infty x_i z^i} \sum_{\ell=0}^{\infty} \mathbf{s}_\ell(-y) \rho_{j+\ell}(z).$$

Lemma 2.2 *In the context of Proposition 1.1, the polynomials above have the following alternative representation in terms of the entries $\mu_{ij} = \langle z^i, \rho_j(z;x,y) \rangle$ of m:*

$$\begin{aligned} p_n^{(1)}(\lambda;x,y) &= \frac{\det \left(\langle z^i, (\lambda - z)\rho_j(z;x,y) \rangle \right)_{0 \le i,j \le n-1}}{\det \left(\langle z^i, \rho_j(z;x,y) \rangle \right)_{0 \le i,j \le n-1}} \\ &= \frac{\det(\lambda \mu_{ij} - \mu_{i+1,j})_{0 \le i,j \le n-1}}{\tau_n(x,y)} \end{aligned} \tag{2.15}$$

$$\begin{aligned} p_n^{(2)}(\lambda;x,y) &= \frac{\det \left(\langle z^i, \lambda \rho_j(z;x,y) - \rho_{j+1}(z;x,y) \rangle \right)_{0 \le i,j \le n-1}}{\det \left(\langle z^i, \rho_j(z;x,y) \rangle \right)_{0 \le i,j \le n-1}} \\ &= \frac{\det(\lambda \mu_{ij} - \mu_{i,j+1})_{0 \le i,j \le n-1}}{\tau_n(x,y)} \end{aligned} \tag{2.16}$$

Proof: The proof follows from the representation (2.11) of $p_n^{(1)}$ above, the representation (1.5) and (1.6) of τ_n, the representation (1.15) of ρ_j and from the following identities:

$$\begin{aligned} \lambda \mu_{ij}(x - [\lambda^{-1}], y) &:= \lambda \left\langle z^i, \rho_j(z; x - [\lambda^{-1}], y) \right\rangle \\ &= \lambda \left\langle z^i \ , \ e^{\sum_1^\infty \left(x_i - \frac{\lambda^{-i}}{i} \right) z^i} \sum_{\ell=0}^{\infty} \mathbf{s}_\ell(-y) \rho_{j+\ell}(z) \right\rangle \\ &= \lambda \left\langle z^i, \left(1 - \frac{z}{\lambda}\right) \rho_j(z;x,y) \right\rangle \\ &= \left\langle z^i, (\lambda - z) \rho_j(z;x,y) \right\rangle \\ &= \lambda \mu_{ij}(x,y) - \mu_{i+1,j}(x,y), \end{aligned}$$

and

$$\begin{aligned}\lambda\mu_{ij}(x,y+[\lambda^{-1}]) &:= \lambda\left\langle z^i,\rho_j(z;x,y+[\lambda^{-1}])\right\rangle\\ &= \lambda\left\langle z^i, e^{\sum_1^\infty x_iz^i}\sum_{\ell=0}^{\infty}\mathbf{s}_\ell(-y-[\lambda^{-1}])\rho_{j+\ell}(z;0,0)\right\rangle\\ &= \left\langle z^i, e^{\sum_1^\infty x_iz^i}\sum_{\ell=0}^{\infty}\left(\lambda\mathbf{s}_\ell(-y)-\mathbf{s}_{\ell-1}(-y)\right)\rho_{j+\ell}(z;0,0)\right\rangle\\ &= \lambda\mu_{ij}(x,y)-\mu_{i,j+1}(x,y),\end{aligned}$$

which is based on the following identity:

$$\begin{aligned}\lambda\sum_0^\infty \mathbf{s}_n(-y-[\lambda^{-1}])\ z^n &= \lambda e^{-\sum_1^\infty(y_i+\frac{\lambda^{-i}}{i})z^i}\\ &= \lambda\sum_0^\infty\mathbf{s}_n(-y)z^n\left(1-\frac{z}{\lambda}\right)\\ &= \sum_0^\infty\left(\lambda\mathbf{s}_n(-y)-\mathbf{s}_{n-1}(-y)\right)z^n.\end{aligned}$$

■

Corollary 2.3 *Given weights $\rho_0,\rho_1,\dots,\rho_{n-1}$, the following identity holds:*

$$\det(\langle z^i,(\lambda-z)\rho_j(z)\rangle)_{0\le i,j\le n-1}=\det\left(\begin{array}{ccc|c}\langle z^0,\rho_0(z)\rangle & \cdots & \langle z^0,\rho_{n-1}(z)\rangle & 1\\ \vdots & & \vdots & \vdots\\ \langle z^n,\rho_0(z)\rangle & \cdots & \langle z^n,\rho_{n-1}(z)\rangle & \lambda^n\end{array}\right)$$

Proof: From Lemma 2.2, it follows that $p_n^{(1)}$ has two alternative expressions (2.13) and (2.15). Equating the two leads to the identity above. ■

Remark: Formula (2.15) and hence (2.13) just depend on the first formula (2.11) and $\tau_n=\det(\mu_{ij})_{0\le i,j\le n-1}$, with $\mu_{ij}(x,y)=\langle z^i,e^{\sum x_iz^i}\rho_j(y,t)\rangle$. The y-dependence is unimportant.

3 From m-periodic weight sequences to $2m+1$-band matrices

Given the m-periodic sequence of weights

$$\rho=(\rho_j)_{j\ge0}=(\rho_0,\rho_1,\dots,\rho_{n-1},z^m\rho_0,\dots,z^m\rho_{m-1},z^{2m}\rho_0,\dots,z^{2m}\rho_{m-1},\dots), \tag{3.1}$$

consider the initial value problem

$$\frac{\partial m_\infty}{\partial x_n} = \Lambda^n m_\infty, \frac{\partial m_\infty}{\partial y_n} = -m_\infty \Lambda^{\top n}, \text{ with initial } m_\infty(0,0) = (\langle z_i, \rho_j \rangle)_{0\le i,j<\infty}, \tag{3.2}$$

and the associated 2-Toda lattice equations

$$\frac{\partial \mathcal{L}_i}{\partial x_n} = [(\mathcal{L}_1^n)_+, \mathcal{L}_i], \quad \frac{\partial \mathcal{L}_i}{\partial y_n} = [(\mathcal{L}_2^n)_-, \mathcal{L}_i]. \tag{3.3}$$

In proposition 1.1, we gave the solution to the initial value problem (3.2) in general, whereas in theorem 3.1, we shall give the solution for m-periodic sequences of weights. This extra-structure will be important, when we deal with Darboux transforms.

Theorem 3.1 *Given the initial m-periodic weights (3.1), the systems of differential equations (3.2) has the following solutions with regard to the time parameters $(\bar{x}, \bar{y}, \bar{t})$, introduced in (2.1):*

$$m_\infty\left(\rho(z;\bar{x},\bar{y},\bar{t})\right) = \left(\left\langle z^i, \rho_j(z;\bar{x},\bar{y},\bar{t})\right\rangle\right)_{0\le i,j<\infty}, \tag{3.4}$$

where

$$\rho_j(z;\bar{x},\bar{y},\bar{t}) := e^{\sum_1^\infty \bar{x}_r z^r} e^{\sum_{\ell=1}^\infty \bar{t}_{\ell m} z^{\ell m}} \sum_{\ell=0}^\infty \mathbf{s}_\ell(-\bar{y})\rho_{j+\ell}(z). \tag{3.5}$$

is an m-periodic sequence of weights. Then the polynomials $p_n^{(1)}$, with $\mu_{ij} := \mu_{ij}(\rho(z;\bar{x},\bar{y},\bar{t}))$ and $\tau_n(\bar{x},\bar{y},\bar{t}) = \det m_n(\rho(z;\bar{x},\bar{y},\bar{t}))$,

$$\begin{aligned} p_n^{(1)}(z;\bar{x},\bar{y},\bar{t}) &= \frac{1}{\tau_n(\bar{x},\bar{y},\bar{t})} \det \left(\begin{array}{ccc|c} \mu_{00} & \dots & \mu_{0,n-1} & 1 \\ \vdots & & \vdots & \vdots \\ \mu_{n-1,0} & \dots & \mu_{n-1,n-1} & z^{n-1} \\ \hline \mu_{n0} & \dots & \mu_{n,n-1} & z^n \end{array} \right) \\ &= \frac{\det(z\mu_{ij} - \mu_{i+i,j})_{0\le i,j\le n-1}}{\tau_n(\bar{x},\bar{y},\bar{t})} \end{aligned}$$

give rise to matrices $L = \mathcal{L}_1^m$, defined by $z^m p^{(1)} = L p^{(1)}$, such that $L = \mathcal{L}_1^m$ is a $2m+1$-band matrix. The matrix $\mathcal{L}_1$ satisfies equations (3.3) and the $2m+1$-band matrix L the m-reduced Toda lattice equations (2.4).

Proof: Since

$$\rho_{j+km} = z^{km}\rho_j, \quad j,k = 0,1,2,\dots,$$

we have

$$\begin{aligned} 0 &= \langle z^i, z^{km}\rho_j - \rho_{j+km} \rangle \\ &= \langle z^{i+km}, \rho_j \rangle - \langle z^i, \rho_{j+km} \rangle \\ &= \mu_{i+km,j} - \mu_{i,j+km} \\ &= \left(\Lambda^{km} m_\infty - m_\infty \Lambda^{\top km}\right)_{ij}, \end{aligned}$$

and so m_∞ satisfies (i) of proposition 2.1 at $(x,y)=(0,0)$ and hence for all (x,y). Therefore, by proposition 2.1, $L:=\mathcal{L}_1^m$ is a $2m+1$ band matrix.

From proposition 1.1, we know that the expression below for m_∞ is a solution of the initial value problem (3.2). The proof of (3.4) follows the lines of calculation (1.16). From there one computes

$$
\begin{aligned}
&m_\infty(\rho(z;x,y))\\
&\quad= e^{\sum_1^\infty x_n\Lambda^n} m_\infty(\rho(z;0,0)) e^{-\sum_1^\infty y_n\Lambda^{\top n}}\\
&\quad= e^{\sum_1^\infty x_n\Lambda^n} \left\langle z^i,\rho_j(z;0,0)\right\rangle_{0\le i,j<\infty} e^{-\sum_{k=1}^\infty y_{km}\Lambda^{\top km}} e^{-\sum_1^\infty \bar y_r\Lambda^{\top r}}\\
&\quad= \sum_0^\infty \mathbf{s}_n(x)\Lambda^n \left\langle z^i,\rho_j(z;0,0)\right\rangle \sum_0^\infty \mathbf{s}_r(-y_m,-y_{2m},\ldots)\Lambda^{\top mr} e^{-\sum_1^\infty \bar y_r\Lambda^{\top r}}\\
&\quad= \left\langle \sum_0^\infty \mathbf{s}_n(x) z^{i+n}, \sum_0^\infty \mathbf{s}_r(-y_m,-y_{2m},\ldots)\rho_{j+rm}(z;0,0)\right\rangle_{0\le i,j<\infty} e^{-\sum_1^\infty \bar y_r\Lambda^{\top r}}\\
&\quad= \left\langle e^{\sum_1^\infty x_r z^r} z^i, \sum_0^\infty \mathbf{s}_r(-y_m,-y_{2m},\ldots) z^{rm}\rho_j(z;0,0)\right\rangle_{0\le i,j<\infty} e^{-\sum_1^\infty \bar y_r\Lambda^{\top r}}\\
&\quad= \left\langle z^i, e^{\sum_1^\infty \bar x_r z^r} e^{\sum_{n=1}^\infty x_{km} z^{km}} e^{-\sum_{k=1}^\infty y_{km} z^{km}} \rho_j(z;0,0)\right\rangle_{0\le i,j<\infty} e^{-\sum_1^\infty \bar y_r\Lambda^{\top r}}\\
&\quad= \left\langle z^i, e^{\sum_1^\infty \bar x_r z^r} e^{\sum_{k=1}^\infty \bar t_{km} z^{km}} \rho_j(z;0,0)\right\rangle_{0\le i,j<\infty} e^{-\sum_1^\infty \bar y_r\Lambda^{\top r}}\\
&\quad= \left\langle z^i, e^{\sum_1^\infty \bar x_r z^r} e^{\sum_{k=1}^\infty \bar t_{km} z^{km}} \rho_j(z;0,0)\right\rangle_{0\le i,j<\infty} \sum_{\ell=0}^\infty \mathbf{s}_\ell(-\bar y)\Lambda^{\top \ell}\\
&\quad= \left\langle z^i, e^{\sum_1^\infty \bar x_r z^r} e^{\sum_{k=0}^\infty \bar t_{km} z^{km}} \sum_{\ell=0}^\infty \mathbf{s}_\ell(-\bar y)\rho_{j+\ell}(z;0,0)\right\rangle_{0\le i,j<\infty},
\end{aligned}
$$

which establishes (3.4). The rest follows from (2.13) (see the last remark of section 2) and Lemma 2.2. ■

In the following we show that m-periodic sequences of weights lead to $2m+1$ band matrices, using a direct proof, thus without invoking the matrices $\mathcal{L}_1$ and $\mathcal{L}_2$ of 2-Toda theory, as in Theorem 3.1. Furthermore, it will be shown that the polynomials $p_n^{(1)}$ are "*orthogonal*" in the sense (3.7). Consider here the slightly more general definition of m-periodic sequences (in comparison to (0.1)):

Definition 3.2 Generalized m-periodic sequences of weights ρ_i *satisfy the following condition: for* $j=0,1,2,\ldots$,

$$z^m\rho_j \in \operatorname{span}\{\rho_0,\ldots,\rho_{m+j}\} \quad \textit{and } z^m\rho_j(z)=c_{j,m+j}\rho_{m+j}(z)+\ldots, \ \textit{with } c_{j,m+j}\neq 0. \tag{3.6}$$

Proposition 3.3 *Given a sequence of weights* $\rho_0(z),\rho_1(z),\ldots$, *the monic polynomials* $p_0(z),p_1(z),\ldots,p_j(z),\ldots$ *of degree* $0,1,2,\ldots$, *defined by*

$$\langle p_i(z),\rho_j(z)\rangle = 0,\quad 0\le j\le i-1 \tag{3.7}$$

are given by the same formula, as in Theorem 3.1, namely

$$p_n(z) = \frac{1}{\det m_n} \det \left(\begin{array}{ccc|c} \mu_{00} & \cdots & \mu_{0,n-1} & 1 \\ \vdots & & \vdots & \vdots \\ \mu_{n-1,0} & \cdots & \mu_{n-1,n-1} & z^{n-1} \\ \hline \mu_{n0} & \cdots & \mu_{n,n-1} & z^n \end{array} \right) \tag{3.8}$$

with $\mu_{ij} = \langle z^i, \rho_j(z) \rangle$, $m_n = \det(\mu_{ij})_{0 \le i,j \le n-1}$. *Moreover, if the* ρ_i *are generalized* m*-periodic, then the polynomials (3.7) satisfy a* $2m+1$*-step relation; i.e., for* $p(z) = (p_0(z), p_1(z), \dots)^\top$,

$$z^m p(z) = Lp(z) \tag{3.9}$$

defines a $2m+1$ *band matrix* L*, with* m *bands above and* m *below the diagonal.*

Proof: For $0 \le k \le n-1$, the inner-product of $p_n(z)$, given by the right hand side of (3.8), with $\rho_k(z)$ automatically vanishes:

$$(\det m_n) \, \langle p_n(z), \rho_k(z) \rangle = \det \left(\langle \mu_{i0}, \mu_{i1}, \dots, \mu_{ik}, \dots, \mu_{i,n-1}, \mu_{ik} \rangle_{i=0,\dots,n} \right) = 0.$$

Furthermore, the orthogonality relation (3.7) determines the monic p_n's uniquely. To prove the second assertion, that L is a $2m+1$ band matrix, we proceed as follows: since $z^m \rho_j(z) = \sum_{r=0}^{m+j} c_{jr} \rho_r(z)$, $j = 0, 1, \dots$, we have

$$\begin{aligned} 0 &= \left\langle z^i, z^m \rho_j - \sum_{r=0}^{m+j} c_{jr} \rho_r(z) \right\rangle, \qquad \text{for all } i, j \ge 0, \\ &= \langle z^{i+m}, \rho_j \rangle - \sum_{r=0}^{m+j} c_{jr} \langle z^i, \rho_r(z) \rangle \\ &= \mu_{m+i,j} - \sum_{r=0}^{m+j} c_{jr} \mu_{ir}, \end{aligned}$$

implying for all $j \ge 0$,

$$\begin{pmatrix} \mu_{m,j} \\ \mu_{m+1,j} \\ \vdots \\ \mu_{m+n,j} \end{pmatrix} = \sum_{r=0}^{m+j} c_{jr} \begin{pmatrix} \mu_{0,r} \\ \mu_{1,r} \\ \vdots \\ \mu_{n,r} \end{pmatrix}.$$

Therefore, by (3.8) the following determinant vanishes for arbitrary $n \ge 0$, as long as $n - 1 \ge m + j$,

$$0 = \frac{1}{D_n(\rho)} \det \left(\begin{array}{ccc|c} \mu_{00} & \cdots & \mu_{0,n-1} & \mu_{mj} \\ \vdots & & \vdots & \vdots \\ & & & \\ \hline \mu_{n,0} & \cdots & \mu_{n,n-1} & \mu_{m+n,j} \end{array} \right) = \langle z^m p_n(z), \rho_j(z) \rangle,$$

for all j such that $0 \le j \le n-m-1$. This implies that

$$\begin{aligned} z^m p_n(z) &\in \{\text{polynomials } q(z) \mid \langle q(z), \rho_j(z)\rangle = 0,\ \text{for } 0 \le j \le n-m-1\}, \\ &= \operatorname{span}\{p_{n-m}(z), p_{n-m+1}(z), \dots, \} \\ &= \operatorname{span}\{p_{n-m}(z), p_{n-m+1}(z), \dots, p_{n+m}(z)\}; \end{aligned}$$

the latter identity is valid, because $z^m p_n(z)$ has degree $n+m$. Therefore L defined by (3.9) is $2m+1$-band, as claimed, ending the proof of Proposition 3.3. ■

Remark: A generalized m-periodic sequence of weights can be transformed in an m-periodic sequence of weights, via an invertible lower-triangular transformation of the ρ_i in the sequence $\rho(z) = (\rho_j(z))_{j\ge 0}$; the new sequence of weights thus obtained become m-periodic, i.e.,

$$z^m \rho_j = z^m \rho_{m+j}. \tag{3.10}$$

Such a transformation leaves the associated polynomials (3.8) unaffected, as is seen from column operations in the defining ratio of determinants in (3.8). These polynomials then lead to $2m+1$ band matrices L, which are thus unaffected by the lower-triangular operations of the ρ_i.

4 Darboux transformations on $2m+1$-band matrices

The vertex operators $\mathbf{X}_i(\lambda) := \mathbf{X}_i(\bar{x}, \bar{y}, \bar{t}; \lambda)$, introduced in the introduction (see [5]), play a central role in this work[12] :

$$\begin{aligned} \mathbf{X}_1(\lambda) &:= \chi(\lambda) e^{\sum_1^\infty \bar{t}_{mi}\lambda^{mi}} e^{-\sum_1^\infty \frac{\lambda^{-mi}}{mi}\frac{\partial}{\partial \bar{t}_{mi}}} e^{\sum_1^\infty \bar{x}_i \lambda^i} e^{-\sum_1^\infty \frac{\lambda^{-i}}{i}\frac{\partial}{\partial \bar{x}_i}} \\ \mathbf{X}_2(\lambda) &:= \chi(\lambda^{-1}) e^{-\sum_1^\infty \bar{t}_{mi}\lambda^{mi}} e^{\sum_1^\infty \frac{\lambda^{-mi}}{mi}\frac{\partial}{\partial \bar{t}_{mi}}} e^{\sum_1^\infty \bar{y}_i \lambda^i} e^{-\sum_1^\infty \frac{\lambda^{-i}}{i}\frac{\partial}{\partial \bar{y}_i}} \Lambda; \end{aligned} \tag{4.1}$$

e.g., $\mathbf{X}_2(\lambda)$ acts on the vector $\tau(\bar{x}, \bar{y}, \bar{t})$, as follows

$$(\mathbf{X}_2(\lambda)\tau(\bar{x}, \bar{y}, \bar{t}))_n = e^{-\sum_1^\infty \bar{t}_{mi}\lambda^{mi}} e^{\sum_1^\infty \bar{y}_i\lambda^i} \lambda^{-n} \tau_{n+1}(\bar{x}, \bar{y} - [\lambda^{-1}], \bar{t} - [\lambda^{-1}])$$

where

$$\begin{aligned} \bar{y} - [\lambda^{-1}] &:= \left(y_1 - \frac{\lambda^{-1}}{1}, \dots, y_{m-1} - \frac{\lambda^{-(m-1)}}{m-1}, 0, y_{m+1} - \frac{\lambda^{-(m+1)}}{m+1}, \dots \dots\right) \\ \bar{t} - [\lambda^{-1}] &:= \left(0, \dots, 0, t_m - \frac{\lambda^{-m}}{m}, 0, \dots, 0, t_{2m} - \frac{\lambda^{-2m}}{2m}, 0, \dots, 0, \dots\right). \end{aligned}$$

The following two theorems were established in [5], and will be applied in section 5 to the concrete τ_n's given by $\tau_n = \det m_n(\rho)$, with the ρ_n's as in (3.5).

[12] $\chi(\lambda) = \operatorname{diag}(\lambda^0, \lambda^1, \lambda^2, \dots)$

Theorem 4.1 (**LU-Darboux transform**) *Given the Toda lattice on semi-infinite $2m+1$-band matrices, each vector $\Phi(\lambda)$ in the m-dimensional null-space, i.e.*[13]*,*

$$\Phi(\lambda)=\frac{\tilde\tau}{\tau}:=\frac{\sum_{k=0}^{m-1}\left(a_k\mathbf{X}_1(\omega^k\lambda)\right)\tau}{\tau}\in(L(t)-\lambda^mI)^{-1}(0,0,...)$$

satisfies, as a function of $\bar x,\bar y,\bar t$, the following equations:

$$L\Phi=\lambda^m\Phi$$

$$\frac{\partial\Phi}{\partial x_i}=(\overline{L^{i/m}})_+\Phi,\qquad\frac{\partial\Phi}{\partial y_i}=(\underline{L^{i/m}})_-\Phi,\qquad\frac{\partial\Phi}{\partial t_{ip}}=(L^i)_+\Phi,\tag{4.2}$$

for $i=1,2,...$ not multiples of m for the x_i and y_i equations. Each $\Phi(\lambda)$ determines an LU-Darboux transform, depending projectively on the $m-1$ parameters a_i; namely

$$L-\lambda^mI\longmapsto\tilde L-\lambda^mI:=(\beta\Lambda^0+\Lambda)(L-\lambda^mI)(\beta\Lambda^0+\Lambda)^{-1}$$

with

$$\beta_n=-\frac{\Phi_{n+1}(\lambda)}{\Phi_n(\lambda)};\tag{4.3}$$

it acts on τ as

$$\tau\longmapsto\tilde\tau=\tau\Phi=\sum_{k=0}^{m-1}\left(a_k\mathbf{X}_1(\omega^k\lambda)\right)\tau.\tag{4.4}$$

Defining $e_i:=(0,...,0,\underbrace{1}_{i},0,...)\in\mathbf{R}^\infty$, as before, we have:

Theorem 4.2 (**UL-Darboux transform**) *Given the Toda lattice on semi-infinite $2m+1$-band matrices, the space $(L-\lambda^mI)^{-1}\operatorname{span}\{e_0,e_1,...,e_m\}$ is $2m$-dimensional and thus depends projectively on $2m-1$ free parameters, i.e.,*

$$\begin{aligned}\Phi(\lambda)=\frac{\Lambda\tilde\tau}{\tau}\quad&:=\quad\frac{\sum_{k=0}^{m-1}\left(a_k\mathbf{X}_1(\omega^k\lambda)+b_ke^{\sum_1^\infty t_{im}\lambda^{im}}\mathbf{X}_2(\omega^k\lambda)\right)\tau}{\tau}\\&\in\quad(L(t)-\lambda^mI)^{-1}\operatorname{span}\{e_0,e_1,...,e_m\}.\end{aligned}$$

The vector $\Phi(\lambda)$, as a function of $\bar x,\bar y,\bar t$, satisfies the same equations (4.2) and determines a UL-Darboux transform, with the same β as (4.3) (but depending projectively on $2m-1$ free parameters):

$$L-\lambda^mI\longmapsto\tilde L-\lambda^mI:=(\Lambda^{-1}\beta+I)(L-\lambda^mI)(\Lambda^{-1}\beta+I)^{-1};$$

it induces a map on τ :

$$\tau\longmapsto\tilde\tau=\Lambda^{-1}(\tau\Phi)=\Lambda^{-1}\sum_{k=0}^{m-1}\left(a_k\mathbf{X}_1(\omega^k\lambda)+b_ke^{\sum_1^\infty t_{im}\lambda^{im}}\mathbf{X}_2(\omega^k\lambda)\right)\tau.$$

[13] ω is a primitive mth root of unity.

5 Proof of Theorems 0.1 and 0.2: Induced Darboux maps on m-periodic weights

In order to prove Theorems 0.1 and 0.2, we apply Theorems 4.1 and 4.2 to the τ-functions given by

$$\tau_n(\bar{x},\bar{y},\bar{t}) = D_n(\rho(z;\bar{x},\bar{y},\bar{t})) := D_n(\rho_0(z;\bar{x},\bar{y},\bar{t}),\rho_1(z;\bar{x},\bar{y},\bar{t}),\dots) = \det m_n(\rho(z;\bar{x},\bar{y},\bar{t}))$$

with

$$\rho_j(z;\bar{x},\bar{y},\bar{t}) = e^{\sum_1^\infty \bar{x}_r z^r} e^{\sum_{\ell=1}^\infty \bar{t}_{\ell m} z^{\ell m}} \sum_{\ell=0}^{\infty} \mathbf{s}_\ell(-\bar{y})\rho_{j+\ell}(z), \tag{5.1}$$

as in (3.5), where the initial condition $\rho(z) = (\rho_j(z))_{j\geq 0}$ forms an m-periodic sequence of weights. We now perform Darboux transformations on $L(\bar{x},\bar{y},\bar{t})$, which satisfies the *$m$-reduced Toda lattice equations* (2.4). Then, in the end, put $\bar{x}=\bar{y}=\bar{t}=0$. Theorems 5.1 and 5.2 are the precise analogues of Theorems 4.1 and 4.2:

Theorem 5.1 *(***LU-Darboux***) The Darboux transform for a semi-infinite $2m+1$-band matrix, generated by the m-periodic sequences of weights $\rho(z;\bar{x},\bar{y},\bar{t})$ above,*

$$L - \lambda^m I \mapsto \tilde{L} - \lambda^m I = (\beta\Lambda^0 + \Lambda)(L - \lambda^m I)(\beta\Lambda^0+\Lambda)^{-1}, \tag{5.2}$$

defines a new $2m+1$-band matrix $\tilde{L}$, provided (ω is a primitive mth root of unity)

$$\beta_n = -\frac{\Phi_{n+1}(\lambda)}{\Phi_n(\lambda)}, \quad \Phi_n(\lambda) = \frac{\sum_{k=0}^{m-1} a_k \mathbf{X}_1(w^k\lambda) D_n(\rho(z;\bar{x},\bar{y},\bar{t}))}{D_n(\rho(z;\bar{x},\bar{y},\bar{t}))} \tag{5.3}$$

<u>Case 1</u> *For the special choice*

$$\Phi_n^{(k)}(\lambda) = a_k \frac{\mathbf{X}_1(w^k\lambda)D_n(\rho(z;\bar{x},\bar{y},\bar{t}))}{D_n(\rho(z;\bar{x},\bar{y},\bar{t}))}$$

with arbitrary, but fixed $1\leq k\leq n$, the Darboux transformation maps $\tau_n(\bar{x},\bar{y},\bar{t}) = D_n(\rho)$ into a D_n associated with a new m-periodic sequence of weights:

$$\begin{aligned} D_n(\rho(z;\bar{x},\bar{y},\bar{t})) \mapsto \tilde{D}_n &= D_n(\rho(z;\bar{x},\bar{y},\bar{t}))\Phi_n^{(k)}(\lambda) \\ &= \tilde{a}_k D_n((\omega^k\lambda - z)\rho(z;\bar{x},\bar{y},\bar{t})) \end{aligned} \tag{5.4}$$

<u>Case 2</u> *A general linear combination*

$$\Phi_n(\lambda) = \frac{\sum_{k=0}^{m-1} a_k \mathbf{X}_1(w^k\lambda) D_n(\rho(z;\bar{x},\bar{y},\bar{t}))}{D_n(\rho(z;\bar{x},\bar{y},\bar{t}))} \tag{5.5}$$

leads to the map

$$\begin{aligned}\tau_n(\bar{x},\bar{y},\bar{t}) = D_n(\rho(z;\bar{x},\bar{y},\bar{t}) \mapsto \tilde{\tau}_n(\bar{x},\bar{y},\bar{t}) &= D_n\left(\rho(z;\bar{x},\bar{y},\bar{t})\right)\Phi_n^{(k)}(\lambda)\\ &= \sum_{k=0}^{m-1}\tilde{a}_k D_n\Big((\omega^k\lambda - z)\rho(z;\bar{x},\bar{y},\bar{t})\Big)\\ &= (-1)^n \det\Big(\langle z^i,\tilde{\rho}_0\rangle,\langle z^i,\tilde{\rho}_1\rangle,\ldots,\langle z^i,\tilde{\rho}_n\rangle\Big)_{0\le i\le n},\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}\tilde{\rho}_0 : &= \sum_{k=0}^{m-1}\tilde{a}_k\delta(z-\omega^k\lambda)\\ \tilde{\rho}_\ell : &= \rho_{\ell-1}(z;\bar{x},\bar{y},\bar{t}), \qquad \textit{for } \ell\ge 1,\end{aligned} \tag{5.7}$$

and

$$\tilde{a}_k = a_k e^{\sum_{i=1}^\infty \bar{t}_{im}\lambda^{im}} e^{\sum_{i=1}^\infty \bar{x}_i(\omega^k\lambda)^i}.$$

Remark: For the general case (case 2), (5.6) is the determinant a $(n+1)\times(n+1)$ matrix, instead of $n\times n$. Therefore, to the best of our knowledge, this τ-function is not generated in the usual way, as a determinant of the $n\times n$ upper-left corner of the moment matrix. If all but one of the a_k's vanish, as in case 1, then the τ-functions are generated in the usual way, as appears immediately from the second identity of (5.4). In the next statement, this problem will be absent.

Theorem 5.2 *(***UL - Darboux***) The Darboux transform for a semi-infinite $2m+1$-band matrix, arising from a m-periodic weights $\rho(z;\bar{x},\bar{y},\bar{t})$,*

$$L-\lambda^m I \longmapsto \tilde{L}-\lambda^m I = (\Lambda^\top\beta+I)(L-\lambda^m I)(\Lambda^\top\beta+I)^{-1}, \tag{5.8}$$

maps L into a new $2m+1$-band matrix $\tilde{L}$, provided (with $D(\rho) := (D_0(\rho), D_1(\rho),\ldots)$),

$$\beta_n = -\frac{\Phi_{n+1}(\lambda)}{\Phi_n(\lambda)}, \quad \Phi_n(\lambda) = \frac{\left(\sum_{k=0}^{m-1}\left(a_k\mathbf{X}_1(\omega^k\lambda) + b_k e^{\sum\bar{t}_{im}\lambda^{im}}\mathbf{X}_2(\omega^k\lambda)\right)D(\rho)\right)_n}{D_n(\rho)}. \tag{5.9}$$

It acts on $\tau_n(\bar{x},\bar{y},\bar{t}) = D_n(\rho(z;\bar{x},\bar{y},\bar{t}))$ as follows

$$\begin{aligned}\tau_n := D_n(\rho(z;\bar{x},\bar{y},\bar{t})) \mapsto \tilde{\tau}_n &= D_{n-1}(\rho(z;\bar{x},\bar{y},\bar{t}))\Phi_{n-1}(\lambda)\\ &= (-1)^{n-1}\det\Big(\langle z^i,\tilde{\rho}_0\rangle,\langle z^i,\tilde{\rho}_1\rangle,\ldots\langle z^i,\tilde{\rho}_{n-1}\rangle\Big)_{0\le i\le n-1}.\end{aligned}$$

with

$$\begin{aligned}\tilde{\rho}_0 : &= \tilde{\rho}_0(z;\bar{x},\bar{y},\bar{t}) := \sum_{k=0}^{m-1}\left(\tilde{a}_k\delta(z-\omega^k\lambda) + \tilde{b}_k\frac{\rho_k(z;\bar{x},\bar{y},\bar{t})}{z^m-\lambda^m}\right)\\ \tilde{\rho}_\ell : &= \rho_{\ell-1}(z;\bar{x},\bar{y},\bar{t}), \qquad \textit{for } \ell\ge 1,\end{aligned} \tag{5.10}$$

where

$$\tilde a_k = a_k e^{\sum_1^\infty \bar x_i(\omega^k\lambda)^i} e^{\sum_1^\infty \bar t_{im}\lambda^{im}}, \quad \tilde b_k = -\lambda^{m-k}\sum_{j=0}^{m-1} b_j e^{\sum_{i\geq 0}\bar y_i(\omega^j\lambda)^i}\omega^{-jk}. \tag{5.11}$$

If $\tilde b_{m-1}\neq 0$, then the $\tilde\rho_0,\tilde\rho_1,\ldots$ form a generalized m-periodic sequence.

<u>*Remark*</u>: Although the new sequence $\tilde\rho(z;\bar x,\bar y,\bar t)$ is generalized m-periodic in the sense of (3.6), it does not lead to a solution m_∞ of the differential equations (3.2); in other words, it only satisfies (3.5) in the $\bar x$ and $\bar t$ variables, but not in the $\bar y$ variable. Of course, the matrix $\tilde L$ remains a $2m+1$-band matrix, since it is effectively constructed from the new polynomials $p_n(z;\bar x,\bar y,\bar t)$, defined by (3.8) with the new ρ's; see remark at the end of section 3.

Corollary 5.3 *An appropriate choice of a_k, and appropriate limits $b_k\mapsto\infty$ and $\lambda\mapsto 0$ in Theorem 5.2 yield the following Darboux transformation on the weights $\rho(z;\bar x;\bar y;\bar t)$:*

$$\rho=(\rho_0,\rho_1,\rho_2,\ldots)\mapsto\tilde\rho=(\tilde\rho_0,\tilde\rho_1,\tilde\rho_2,\ldots),$$

where

$$\begin{aligned}\tilde\rho_0 &= \sum_{k=0}^{m-1}\left(c_k\left(\frac{d}{dz}\right)^k\delta(z)+d_k\frac{\rho_k(z;\bar x;\bar y;\bar t)}{z^m}\right), \quad d_{m-1}\neq 0,\\ \tilde\rho_\ell &= \rho_{\ell-1}(z;\bar x,\bar y,\bar t).\end{aligned} \tag{5.12}$$

Before proving theorems 5.1 and 5.2 and corollary 5.3, we need a crucial Lemma :

Lemma 5.4 *The following two identities hold for the m-periodic sequences of weights of (5.1):*

$$\begin{aligned}&\mathbf{X}_1(\lambda)D_n(\rho)\\ &\quad= e^{\sum_1^\infty \bar t_{im}\lambda^{im}}e^{\sum_1^\infty \bar x_i\lambda^i}D_n((\lambda-z)\rho)\\ &\quad= e^{\sum_1^\infty \bar t_{im}\lambda^{im}}e^{\sum_1^\infty \bar x_i\lambda^i}(-1)^n\det(\langle z^i,\delta(z-\lambda)\rangle,\langle z^i,\rho_0\rangle,\ldots,\langle z^i,\rho_{n-1}\rangle)_{0\leq i\leq n},\end{aligned} \tag{5.13}$$

$$\begin{aligned}&\Lambda^{-1}e^{\sum_{i=1}^\infty \bar t_{im}\lambda^{im}}\mathbf{X}_2(\lambda)D_n(\rho)\\ &\quad= e^{\sum_1^\infty \bar y_i\lambda^i}(-1)^{n-1}\\ &\qquad\det\left(\left\langle z^i,\frac{\sum_{r=0}^{m-1}\lambda^{m-r}\rho_r}{\lambda^m-z^m}\right\rangle,\langle z^i,\rho_0\rangle,\ldots,\ldots\langle z^i,\rho_{n-2}\rangle\right)_{0\leq i\leq n-1},\end{aligned} \tag{5.14}$$

with all the ρ_j's in the determinants above evaluated at $\bar x,\bar y,\bar t$ according to formula (5.1).

<u>*Proof*</u>: : Here we use the first solution m_∞ of (3.4) (and its calculation in the proof of Theorem 3.1), and in the second equality, we use the familiar formula $e^{-\sum u^i/i}=1-u$. So, one computes, using $\mathbf{X}_1(\lambda)$, as defined in (4.1):

$$
\begin{aligned}
&\mathbf{X}_1(\lambda)D_n\Big(\rho(z;\bar x,\bar y,\bar t)\Big)\\
&= \lambda^n e^{\sum_{\ell=1}^{\infty}\bar t_{\ell m}\lambda^{\ell m}} e^{\sum_1^\infty \bar x_i\lambda^i} e^{-\sum_1^\infty \frac{\lambda^{im}}{im}\frac{\partial}{\partial t_{im}}} e^{-\sum\frac{\lambda^{-i}}{i}\frac{\partial}{\partial \bar x_i}}\\
&\qquad \cdot \det\left\{(\langle z^i,\rho_j(z;0,0,0)e^{\sum \bar x_r z^r} e^{\sum_{\ell=1}^\infty \bar t_{\ell m} z^{\ell m}}\rangle)_{0\le i,j\le\infty} e^{-\sum_1^\infty \bar y_r \Lambda^{r\top}}\right\}_{0\le i,j\le n-1}\\
&= \lambda^n e^{\sum_{\ell=1}^{\infty}\bar t_{\ell m}\lambda^{\ell m}} e^{\sum \bar x_i\lambda^i}\\
&\qquad \det\left\{\left(\left\langle z^i,\rho_j(z,0,0,0)e^{-\sum_1^\infty\frac{1}{r}(\frac{z}{\lambda})^r} e^{\sum_1^\infty \bar x_r z^r} e^{\sum_1^\infty \bar t_{\ell m} z^{\ell m}}\right\rangle\right)_{0\le i,j\le\infty} e^{-\sum_1^\infty \bar y_r\Lambda^{r\top}}\right\}_{0\le i,j\le n-1}\\
&= e^{\sum_{\ell=1}^{\infty}\bar t_{\ell m}\lambda^{\ell m}} e^{\sum \bar x_i\lambda^i} D_n\Big((\lambda-z)\rho(z;\bar x,\bar y,\bar t)\Big),
\end{aligned}
$$

upon bringing λ^n in the $n\times n$ determinant, and using again the first expression (3.4) for m_∞. But, using (1.5) and (1.14), we compute, where in this calculation $\rho_i := \rho_i(\bar x,\bar y,\bar t)$,

$$
\begin{aligned}
D_n\left((\lambda-z)\rho\right) &= \det(\left\langle z^i,(\lambda-z)\rho_0\right\rangle,\dots,\left\langle z^i,(\lambda-z)\rho_{n-1}\right\rangle)_{0\le i\le n-1}\\
&= \det\Big(\langle z^i,\rho_0\rangle,\dots,\langle z^i,\rho_{n-1}\rangle,\lambda^i\Big)_{0\le i\le n},\ \text{ using Corollary 2.3,}\\
&= (-1)^n\det\Big(\left\langle z^i,\delta(z-\lambda)\right\rangle,\langle z^i,\rho_0\rangle,\dots,\langle z^i,\rho_{n-1}\rangle\Big)_{0\le i\le n},
\end{aligned}
$$

using the δ-function property, thus establishing the identity (5.13).

For future use, we shall need the following easy identities :

$$
e^{\sum_{i=1}^\infty\frac{a^{im}}{im}} = e^{\frac{1}{m}\sum_1^\infty\frac{(a^m)^i}{i}} = (\frac{1}{1-a^m})^{1/m}, \tag{5.15}
$$

and (in the exponential, one sums over i's, not multiples of m)

$$
\begin{aligned}
e^{\sum_{\substack{m\nmid i\\ i=1}}^\infty\frac{a^i}{i}} &= e^{\sum_1^\infty\frac{a^i}{i}}e^{-\sum_1^\infty\frac{a^{im}}{im}}\\
&= \frac{(1-a^m)^{1/m}}{1-a}\\
&= \frac{1-a^m}{1-a}(1-a^m)^{-1+1/m}\\
&= \sum_0^{m-1} a^i(1-a^m)^{-1+1/m}.
\end{aligned}\tag{5.16}
$$

Notice that, for any moment matrix m_∞ defined by m-periodic weights,

$$
\left(m_\infty\left(\frac{\Lambda^\top}{\lambda}\right)^n\right)_{ij} = \frac{\mu_{i,j+n}}{\lambda^n} = \left\langle z^i,\frac{\rho_{j+n}}{\lambda^n}\right\rangle;
$$

in particular, using the periodicity of the sequence $\rho_j=\rho_j(z;0,0,0)$, we have

$$
\left(m_\infty\left(\frac{\Lambda^\top}{\lambda}\right)^{mk}\right)_{ij} = \left\langle z^i,\frac{\rho_{j+mk}}{\lambda^{mk}}\right\rangle = \left\langle z^i,\left(\frac{z}{\lambda}\right)^{mk}\rho_j\right\rangle.
$$

Combining these two facts, we find

$$\left(m_\infty\left(\frac{\Lambda^\top}{\lambda}\right)^r f\left(\left(\frac{\Lambda^\top}{\lambda}\right)^m\right)\right)_{ij}=\left\langle z^i, f\left(\left(\frac{z}{\lambda}\right)^m\right)\frac{\rho_{j+r}}{\lambda^r}\right\rangle. \tag{5.17}$$

Now using $\mathbf{X}_2(\lambda)$, defined in (4.1) and using (3.4) for m_∞, compute:

$$\begin{aligned}
&\Lambda^{-1}e^{\sum_{i=1}^\infty t_{im}\lambda^{im}}\mathbf{X}_2(\lambda)D_n\Big(\rho(z,\bar x,\bar y,\bar t)\Big)\\
&= \lambda^{1-n}e^{\sum \bar y_i\lambda^i}e^{\sum_1^\infty \frac{\lambda^{-rm}}{rm}\frac{\partial}{\partial \bar t_{rm}}}e^{-\sum_1^\infty\frac{\lambda^{-r}}{r}\frac{\partial}{\partial \bar y_r}}\\
&\qquad \det\left\{\left\langle z^i,\rho_j(z;0,0,0)e^{\sum_{r=1}^\infty \bar t_{rm}z^{rm}}e^{\sum \bar x_r z^r}\right\rangle_{0\le i,j<\infty}e^{-\sum \bar y_r\Lambda^{Tr}}\right\}_{0\le i,j\le n-1}\\
&= \lambda^{1-n}e^{\sum \bar y_i\lambda^i}\det\Bigg\{\left\langle z^i,\rho_j(z;0,0,0)e^{\sum_{r=1}^\infty\frac{1}{\ell m}\left(\frac{z}{\lambda}\right)^{\ell m}}e^{\sum_{r=1}^\infty \bar t_{rm}z^{rm}}e^{\sum \bar x_r z^r}\right\rangle_{0\le i,j<\infty}\\
&\qquad\qquad e^{\sum_{m\nmid r}\frac{1}{r}\left(\frac{\Lambda^\top}{\lambda}\right)^r}e^{-\sum \bar y_r\Lambda^{\top r}}\Bigg\}_{0\le i,j<n-1}\\
&= \lambda^{1-n}e^{\sum \bar y_i\lambda^i}\\
&\qquad \det\Bigg\{\left\langle z^i\ ,\ e^{\sum_1^\infty \bar t_{rm}z^{rm}}e^{\sum \bar x_r z^r}\frac{\rho_j(z;0,0,0)}{(1-(z/\lambda)^m)^{1/m}}\right\rangle_{0\le i,j<\infty}\\
&\qquad\qquad \frac{\sum_0^{m-1}\left(\frac{\Lambda^\top}{\lambda}\right)^i}{\left(1-\left(\frac{\Lambda^\top}{\lambda}\right)^m\right)^{1-1/m}}e^{-\sum_1^\infty \bar y_r\Lambda^{\top r}}\Bigg\}_{0\le i,j\le n-1},\ \text{using (5.15) and (5.16),}\\
&= \lambda^{1-n}e^{\sum \bar y_i\lambda^i}\\
&\qquad \det\Bigg\{\left\langle z^i,e^{\sum_1^\infty \bar t_{rm}z^{rm}}e^{\sum \bar x_r z^r}\frac{\sum_{r=0}^{m-1}\frac{\rho_{j+r}(z;0,0,0)}{\lambda^r}}{\left(1-\left(\frac{z}{\lambda}\right)^m\right)^{1/m}\left(1-\left(\frac{z}{\lambda}\right)^m\right)^{1-1/m}}\right\rangle_{0\le i,j<\infty}\\
&\qquad\qquad e^{-\sum \bar y_r\Lambda^{\top r}}\Bigg\}_{0\le i,j\le n-1},\ \text{using (5.17),}\\
&= \lambda^{1-n}e^{\sum \bar y_i\lambda^i}\det\Bigg\{\left\langle z^i,e^{\sum_1^\infty \bar t_{rm}z^{rm}}e^{\sum \bar x_r z^r}\frac{\sum_{r=0}^{m-1}\lambda^{m-r}\rho_{j+r}(z;0,0,0)}{\lambda^m-z^m}\right\rangle_{0\le i,j<\infty}\\
&\qquad\qquad e^{-\sum \bar y_r\Lambda^{\top r}}\Bigg\}_{0\le i,j\le n-1}\\
&= \lambda e^{\sum \bar y_i\lambda^i}\det\left\{\left\langle z^i,\sum_{r=0}^{m-1}\frac{\lambda^{m-1-r}}{\lambda^m-z^m}\rho_{j+r}(z,\bar x,\bar y,\bar t)\right\rangle_{0\le i,j\le n-1}\right\}\\
&= \lambda e^{\sum \bar y_i\lambda^i}(-1)^{n-1}\det\Bigg(\left\langle z^i,\sum_{r=0}^{m-1}\frac{\lambda^{m-1-r}\rho_r(z;\bar x,\bar y,\bar t)}{\lambda^m-z^m}\right\rangle,\left\langle z^i,\rho_0(z;\bar x,\bar y,\bar t)\right\rangle,
\end{aligned}$$

$$\left.\cdots,\langle z^i,\rho_{n-2}(z;\bar x,\bar y,\bar t)\rangle\right)_{0\le i\le n-1}.$$

The second from the last expression is a consequence of (3.4) and (3.5), according to the argument in the proof of Theorem 3.1 and the linearity of (3.5) with respect to the measures $\rho=(\rho_0,\rho_1,...)$, while the last line is obtained by replacing the jth column C_j by $C_j-\lambda C_{j-1}, 2\le j\le n$, in the previous determinant and using the identity:

$$\begin{aligned}\sum_{r=0}^{m-1}\frac{\lambda^{m-1-r}\rho_{j+r}}{\lambda^m-z^m}-\lambda\sum_{r=0}^{m-1}\frac{\lambda^{m-1-r}\rho_{j+r-1}}{\lambda^m-z^m}&=\frac{\rho_{j+m-1}-\lambda^m\rho_{j-1}}{\lambda^m-z^m}\\&=\frac{z^m\rho_{j-1}-\lambda^m\rho_{j-1}}{\lambda^m-z^m}\\&=-\rho_{j-1}\end{aligned}$$

■

Proof of Theorem 5.1 From theorem 4.1 (the map (4.4)), and from (5.13) of Lemma 5.4, it follows that

$$\begin{aligned}\tau_n=D_n(\rho)\mapsto\tilde\tau_n&=\sum_{k=0}^{m-1}a_k\mathbf{X}_1(\omega^k\lambda)D_n(\rho)\\&=\sum_{k=0}^{m-1}e^{\sum_{1=i}^\infty\bar t_{im}\lambda^{im}}e^{\sum_{i=1}^\infty\bar x_i(\omega^k\lambda)^i}a_kD_n((\omega^k\lambda-z)\rho)\\&=\sum_{k=0}^{m-1}\tilde a_k\ D_n\left((\omega^k\lambda-z)\rho\right)\\&=(-1)^n\sum_{k=0}^{m-1}\tilde a_k\det\left(\langle z^i,\delta(z-\omega^k\lambda)\rangle,\langle z^i,\rho_0\rangle,\cdots,\langle z^i,\rho_{n-1}\rangle\right)_{0\le i\le n}.\end{aligned}$$

The expression on the right hand side of the third identity establishes the second identity (5.6), whereas the last identity establishes the third (5.6), ending the proof of Case 1. Setting all but one $a_k=0$, establishes (5.4) in Case 1. ■

Proof of Theorem 5.2 : According to Theorem 4.2 and Lemma 5.4, the UL-Darboux transform (5.8) with β_n, given in (5.9) acts on $\tau_n(z;\bar x,\bar y,\bar t):=D_n(\rho(z;\bar x,\bar y,\bar t))$ as follows:

$$\begin{aligned}\tau_n\longmapsto\ &\tilde\tau_n\\=\ &\left(\Lambda^{-1}\tau\Phi(\lambda)\right)_n\\=\ &\left(\sum_{k=0}^{m-1}\left(a_k\Lambda^{-1}\mathbf{X}_1(\omega^k\lambda)+b_k\Lambda^{-1}e^{\sum_{i=0}^\infty\bar t_{im}\lambda^{im}}\mathbf{X}_2(\omega^k\lambda)\right)\tau\right)_n\\=\ &(-1)^{n-1}\det\left(\langle z^i,\sum_{k=0}^{m-1}\tilde a_k\delta(z-\omega^k\lambda)\rangle,\langle z^i,\rho_0\rangle,\cdots,\langle z^i,\rho_{n-2}\rangle\right)_{0\le i\le n-1}\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{n-1}\det\Bigg(\langle z^i,\sum_{k=0}^{m-1}b_k'\sum_{r=0}^{m-1}\frac{(\omega^k\lambda)^{m-r}}{\lambda^m-z^m}\rho_r\rangle,\\
&\qquad\qquad\qquad\langle z^i,\rho_0\rangle,\cdots,\langle z^i,\rho_{n-2}\rangle\Bigg)_{0\le i\le n-1}\\
&\text{with } \tilde a_k \text{ as in (5.11) and } b_k'=b_ke^{\sum_1^\infty \bar y_i(\omega^k\lambda)^i},\\
=\;&(-1)^{n-1}\det\Bigg(\bigg\langle z^i,\sum_{k=0}^{m-1}\tilde a_k\delta(z-\omega^k\lambda)+\sum_{r=0}^{m-1}\frac{\lambda^{m-r}}{\lambda^m-z^m}\left(\sum_{k=0}^{m-1}b_k'\omega^{-kr}\right)\rho_r\bigg\rangle,\\
&\qquad\qquad\qquad\langle z^i,\rho_0\rangle,\cdots,\langle z^i,\rho_{n-2}\rangle\Bigg)_{0\le i\le n-1}\\
=\;&(-1)^{n-1}\det\Big(\langle z^i,\tilde\rho_0\rangle,\langle z^i,\tilde\rho_1\rangle,\cdots,\langle z^i,\tilde\rho_{n-1}\rangle\Big)_{0\le i\le n-1},
\end{aligned}
$$

using the new $\tilde\rho_i$ defined in (5.10).

Finally, using the δ-function property in the second identity, and using $\tilde\rho_k=\rho_{k-1}$ for k not a multiple of m, we prove the new sequence is generalized m-periodic:

$$
\begin{aligned}
z^m\tilde\rho_0 &= \sum_{k=0}^{m-1}\left(\tilde a_kz^m\delta(z-\omega^k\lambda)+\tilde b_k\frac{\lambda^m+(z^m-\lambda^m)}{z^m-\lambda^m}\rho_k(z)\right)\\
&= \lambda^m\sum_{k=0}^{m-1}\left(\tilde a_k\delta(z-\omega^k\lambda)+\tilde b_k\frac{\rho_k(z)}{z^m-\lambda^m}\right)+\sum_{k=0}^{m-1}\tilde b_k\rho_k(z)\\
&= \lambda^m\tilde\rho_0(z)+\sum_{k=1}^{m}\tilde b_{k-1}\tilde\rho_k(z)\\
&\in \text{ span }\{\tilde\rho_0,\cdots,\tilde\rho_m\},\quad\text{with the condition that } \tilde b_{m-1}\neq 0,\\
&\\
z^m\tilde\rho_k &= z^m\rho_{k-1}=\rho_{k-1+m}=\tilde\rho_{k+m},\quad\text{for } k\ge 1,\ \text{not a multiple of } m,
\end{aligned}
$$

establishing Theorem 5.2. ■

Remark: As already pointed out in the remark following the statement of Theorem 5.2, although the sequence $\rho(\bar x,\bar y,\bar t)$ is generalized m-periodic in the sense of definition 3.2, it is not m-periodic in the sense of (0.1) and it only leads to a solution m_∞ of (3.2) in the $\bar x$ and $\bar t$ variables, but not in $\bar y$. However, since the matrix $\tilde L$ is computed from the new polynomials $p_n(z;\bar x,\bar y,\bar t)$ (defined in Theorem 3.1), by $z^mp=\tilde Lp$ and since establishing the form of p_n only depended on the x-dependence of τ through $\rho(\bar x,\bar y,\bar t)$, it is indeed defined by m-periodic weights.

Proof of Corollary 5.3: The proof follows at once from theorem 5.2 by letting $\lambda\to 0$, and $b_k\to\infty$, and by picking appropriate a_k. ■

Proof of Theorem 0.1, 0.2 and Corollary 0.3 : The proofs follow from setting $(\bar x,\bar y,\bar t)=(0,0,0)$ in Theorems 5.1, 5.2, and Corollary 5.3.

6 Example 1: Darboux transform for tridiagonal matrices

In this section, we specialize to the case $m = 1$, which leads naturally to orthogonal polynomials, to three-step relations, and so to semi-infinite tridiagonal matrices L. The LU-Darboux transform on such matrices consists of decomposing the matrices $L-\lambda I$ as a product of lower- and upper-triangular matrices and multiplying them in the opposite order. The UL-Darboux goes the other way around. Unlike the case of bi-infinite matrices, the LU-Darboux map for the semi-infinite case is a unique operation, of course depending on the parameter λ, whereas UL-Darboux depends on a free parameter σ, besides λ.

What is the effect of this operation on weights? Theorems 5.1 and 5.2 show that LU-Darboux has the effect of multiplying the weight $\rho(z)$ with $\lambda - z$ and UL-Darboux divides the weight by $\lambda - z$, augmented by a delta-function $(\sigma/\lambda)\delta(z-\lambda)$ involving the free parameter σ.

In [5], we have shown that, upon letting the tridiagonal, bi-infinite matrices flow according to the standard Toda lattice, the LU- or UL-Darboux transforms act on the eigenvectors as discrete Wronskians and on the τ-functions as vertex operators especially taylored to the Toda lattice. Both transforms depend on one free (projective) parameter. The reduction to the semi-infinite case cuts out this freedom for the LU-transform, but not for the UL-transform.

This vertex operators technology can be used very efficiently to get the results, after setting $t = 0$; in fact one can establish a dictionary between the three points of view: *weights, vertex operators and Darboux transforms*, as summarized in (0.23); the point of the dictionary is contained in the subsequent theorems and corollaries. The relationship rests on an elementary *addition formula*; namely, the sum of moment determinants D_n and D_{n-1} with regard to specific weights is again a moment determinant D_n, but with respect to a new weight:

$$D_n(\rho) + cD_{n-1}\left((\lambda - z)^2\rho(z)\right) = D_n(\rho(z) + c\delta(\lambda - z));$$

this fact is not surprising, in view of the fact that if the $\tau = (\tau_n)_{n\geq 0}$ is a vector of τ-functions for the standard Toda lattice, then the following expressions

$$\tau(t) + c\mathbf{X}(t,\lambda)\tau(t)$$

forms a Toda τ-vector as well, where $\mathbf{X}(t,\lambda)$ is the standard Toda vertex operator, defined in (0.21), and acting on τ as in (0.22).

An arbitrary weight $\rho(z)$ on $\mathbf{R}$ yields a 1-periodic sequence $(\rho(z), z\rho(z), z^2\rho(z), \ldots)$ and a moment matrix m_∞, satisfying $\Lambda m_\infty = m_\infty \Lambda^\top$ (Hänkel matrix). Also

$$m_n(\rho) = (\mu_{i+j}(\rho))_{0\leq i,j\leq n-1}, \quad D_n(\rho) = \det m_n(\rho), \text{ with } \mu_k(\rho) = \int_{\mathbf{R}} z^k \rho(z)dz, \tag{6.1}$$

with $D_0 = 1$. The orthogonality relations (3.7) lead to monic orthogonal polynomials in z of degree n

$$p_n(z) = \frac{1}{D_n(\rho)} \det \left(\begin{array}{ccc|c} \mu_0(\rho) & \dots & \mu_{n-1}(\rho) & 1 \\ \vdots & & \vdots & \vdots \\ \mu_{n-1}(\rho) & \dots & \mu_{2n-2}(\rho) & z^{n-1} \\ \hline \mu_n(\rho) & \dots & \mu_{2n-1}(\rho) & z^n \end{array} \right), \quad \text{with} \quad \langle p_i, p_j \rho \rangle = \delta_{ij} h_i \quad (6.2)$$

In turn, the semi-infinite vector of polynomials $p = (p_n(z))_{n\geq 0}$ leads to a semi-infinite tridiagonal matrix L, defined by

$$zp = Lp, \text{ with } L = \begin{pmatrix} b_0 & 1 & \\ a_0 & b_1 & \ddots \\ & \ddots & \ddots \end{pmatrix}. \tag{6.3}$$

Theorem 6.1 *(i) Given the weight $\rho(z)$ and $\lambda \in \mathbf{C}$, the eigenvector of L, corresponding to the eigenvalue λ,*

$$(\Phi_n(\lambda))_{n\geq 0} = (p_n(\lambda))_{n\geq 0} = \left(\frac{D_n((\lambda - z)\rho(z))}{D_n(\rho)} \right)_{n\geq 0} \in (L - \lambda I)^{-1}(0, 0, 0, ...) \tag{6.4}$$

specifies a unique LU-Borel factorization

$$L - \lambda I = L_- L_+ = \begin{pmatrix} 1 & 0 & \\ \alpha_0 & 1 & \ddots \\ & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \beta_0 & 1 & \\ 0 & \beta_1 & \ddots \\ & \ddots & \ddots \end{pmatrix},$$

with

$$\beta_n := -\frac{\Phi_{n+1}(\lambda)}{\Phi_n(\lambda)}, \qquad \alpha_{n-1} = b_n - \beta_n - \lambda. \tag{6.5}$$

The LU-Darboux transform

$$L - \lambda = L_- L_+ \longmapsto \tilde{L} - \lambda = L_+ L_-, \tag{6.6}$$

induces the following map on weights $\rho(z)$:

$$\rho(z) \longmapsto \rho(z)(\lambda - z) \tag{6.7}$$

(ii) The two-dimensional eigenspace, corresponding to the eigenvalue λ and with a different boundary condition at $n = 0$, is given by

$$(\Phi_n(\lambda))_{n\geq 0} = \left(\frac{\frac{\sigma}{\lambda} D_n((\lambda - z)\rho(z)) + D_{n+1}\left(\frac{\rho(z)}{\lambda - z}\right)}{D_n(\rho)} \right)_{n\geq 0} \in (L - \lambda I)^{-1}(1, 0, 0, ...). \tag{6.8}$$

It specifies a σ-dependent family of UL-Borel factorizations,

$$L-\lambda = L'_+L'_- = \begin{pmatrix} \alpha_{-1} & 1 & \\ 0 & \alpha_0 & \ddots \\ & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & \\ \beta_0 & 1 & \ddots \\ & \ddots & \ddots \end{pmatrix}, \tag{6.9}$$

with the same β_n and α_{n-1} as in (6.5), but with Φ_n defined by (6.8). This defines UL-Darboux transforms

$$L-\lambda = L'_+L'_- \longmapsto \tilde{L}' - \lambda = L'_-L'_+, \tag{6.10}$$

inducing the following map on weights $\rho(z)$:

$$\rho(z) \longmapsto \left(\frac{\rho(z)}{\lambda - z} + \frac{\sigma}{\lambda}\delta(\lambda - z)\right), \tag{6.11}$$

<u>*Proof*</u>: These statements follow immediately from setting $m=1$ in Theorems 0.1 and 0.2.

Corollary 6.2 *Consider the map $L \longmapsto L''$, defined by a UL-Darboux transform followed by a LU-transform:*

$$L-\lambda = L_+L_- \longmapsto L' - \lambda := L_-L_+ \longmapsto L' - \mu = L'_-L'_+ \longmapsto L'' - \mu := L'_+L'_-,$$

where the parameter of the first UL-Darboux map is given by

$$\sigma := \frac{c\mu}{\mu - \lambda};$$

then, upon taking the limit $\mu \longrightarrow \lambda$, the map above induces a map of weights

$$\rho(z) \longmapsto \rho(z) + c\delta(\lambda - z).$$

Corollary 6.3 *Concatenating m LU-Darboux transforms with parameter μ_i and n UL-Darboux transforms with n_i parameters converging to λ_i ($n_1 + ... + n_r = n$), induces a map of weights:*

$$\rho(z) \longmapsto \left(\frac{\prod_1^m (z-\mu_i)}{\prod_1^r (z-\lambda_k)^{n_k}}\rho(z) + \sum_{k=1}^r \sum_{j=1}^{n_k} c_{kj} \left(\frac{\partial}{\partial z}\right)^{j-1} \delta(z-\lambda_k)\right).$$

Upon picking the μ_i appropriately, the fraction in front of $\rho(z)$ in the formula above disappears.

These statements are established by letting the moment matrix m_∞ flow according to (1.1), and then letting the associated tridiagonal matrix L flow according to the standard Toda lattice (remember $(L^n)_+$ denotes the strictly upper-triangular part of L^n)

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n = 1, 2, \dots . \tag{6.12}$$

In the 3-reduction of 2-Toda, only one set of times $t = \bar{t} = (t_1, t_2, \dots)$ of (2.1) remain. The $(\bar{x}, \bar{y}, \bar{t})$ evolution (3.5) of the weight $\rho(z)$ reduces to the simple formula

$$\rho_t(z) := e^{\sum_1^\infty t_i z^i} \rho(z),$$

which was shown in a direct way in [2], for instance;in other terms, the Toda vector fields (6.12) linearize at the level of the weight $\rho_t(z)$. The deformations $\rho_t(z)$ of $\rho(z)$ enable one to define t-dependent moments $\mu_k(\rho_t(z))$, associated moment matrices $m_n(\rho_t(z))$, and t-dependent monic orthogonal polynomials $p_n(z; t)$ of degree n, with L^2-norms

$$h_n(t) := \int_{\mathbf{R}} p_n^2(t, z) \rho(t, z) dz = \frac{\tau_{n+1}(t)}{\tau_n(t)}. \tag{6.13}$$

The entries of the t-dependent L-matrix are expressed in terms of the τ-functions

$$D_n(\rho_t) = \det m_n(\rho_t) =: \tau_n(t), \tag{6.14}$$

as follows,

$$b_k = \frac{\partial}{\partial t_1} \log \frac{\tau_{k+1}}{\tau_k} \quad \text{and} \quad a_{k-1} = \frac{\tau_{k-1}\tau_{k+1}}{\tau_k^2}. \tag{6.15}$$

Setting $m = 1$ in the *vertex operators* $\mathbf{X}_1(t, \lambda)$ and $\mathbf{X}_2(t, \lambda)$ of (4.1) leads to

$$\mathbf{X}_1(t, \lambda) := \chi(\lambda) X(t, \lambda) \quad \text{and} \quad \mathbf{X}_2(t, \lambda) := \chi(\lambda^{-1}) X(-t, \lambda) \Lambda. \tag{6.16}$$

They are generating functions of symmetries of the standard Toda Lattice and act on τ-vectors; see [5]. The vertex operator $\mathbf{X}(t, \lambda)$, defined in (0.21), is obtained from $\mathbf{X}_1(t, \lambda)$ and $\mathbf{X}_2(t, \lambda)$, as follows

$$\mathbf{X}(t, \lambda) := \lim_{\mu \to \lambda} \frac{1}{1 - \lambda/\mu} \left(e^{\sum t_i \mu^i} \mathbf{X}_2(t, \mu) \right)^{-1} \mathbf{X}_1(t, \lambda) = \Lambda^{-1} \chi(\lambda^2) e^{\sum t_i \lambda^i} e^{-2 \sum \frac{\lambda^{-i}}{i} \frac{\partial}{\partial t_i}} \tag{6.17}$$

has the surprising property (in view of the non-linearity of the problem) that, given a vector $\tau = (\tau_0, \tau_1, \dots)$ of Toda τ-functions, the new vector (see (0.22))

$$\tau + \mathbf{X}(t, \lambda) \tau \tag{6.18}$$

is a new vector of Toda τ-functions. For connections with vertex operator algebras, see V. Kac [17].

The following statements, Theorem 6.4 and Corollary 6.5 are completely parallel with Theorem 6.1 and Corollary 6.2. They provide a *dictionary*, between the three points of view:

Theorem 6.4 *(i) The eigenvector*[14]

$$\begin{aligned}\Phi(t,\lambda):=\frac{\mathbf{X}_1(t,\lambda)\tau(t)}{\tau(t)} &= e^{\sum_0^\infty t_i\lambda^i}\left(\frac{D_n((\lambda-z)\rho_t(z))}{D_n(\rho_t)}\right)_{n\geq 0}\\ &\in (L(t)-\lambda I)^{-1}(0,0,0,...)\end{aligned}\tag{6.19}$$

induces a LU-Borel factorization, as in (6.5), with

$$\alpha_n=\frac{\partial}{\partial t_1}\log\Phi_{n+1}(t,\lambda)-\lambda$$

and

$$\beta_n=-\frac{\Phi_{n+1}(t,\lambda)}{\Phi_n(t,\lambda)}=-\frac{\partial}{\partial t_1}\log\left(\frac{\tau_n}{\tau_{n+1}}\Phi_n(t,\lambda)\right);\tag{6.20}$$

the LU-Darboux transform $L(t)-\lambda\mapsto\tilde L(t)-\lambda$ with new entries $\tilde b_n$ and $\tilde a_n$, is given by (6.6) in terms of the new τ-function:

$$\tau\longmapsto\tilde\tau=\tau\Phi=\mathbf{X}_1(t,\lambda)\tau(t).\tag{6.21}$$

(ii) The eigenvectors

$$\begin{aligned}\Phi(t,\lambda): &= \frac{1}{\lambda}\frac{\left(\sigma\,\mathbf{X}_1(t,\lambda)+\ e^{\sum t_i\lambda^i}\mathbf{X}_2(t,\lambda)\right)\tau(t)}{\tau(t)}\\ &= \left(\frac{\frac{\sigma}{\lambda}e^{\sum t_i\lambda^i}D_n((\lambda-z)\rho_t(z))+D_{n+1}\left(\frac{\rho_t(z)}{\lambda-z}\right)}{D_n(\rho_t)}\right)_{n\geq 0}\\ &\in (L-\lambda I)^{-1}(1,0,0,...)\end{aligned}\tag{6.22}$$

induce a UL-factorization with α and β as in (6.20), but with $\Phi_n(t,z)$ defined in (6.22); it defines a UL-Darboux transform $L(t)-\lambda\mapsto\tilde L'(t)-\lambda$, as in (6.10), with new entries $\tilde b'_n$ and $\tilde a'_n$, given by (6.15) in terms of the new τ-function

$$\tau\longmapsto\tilde\tau'=\Lambda^{-1}\lambda\tau\Phi=\Lambda^{-1}\left(\sigma\ \mathbf{X}_1(t,\lambda)+\ e^{\sum t_i\lambda^i}\mathbf{X}_2(t,\lambda)\right)\tau(t).\tag{6.23}$$

Corollary 6.5 *Consider the map $L(t)\longmapsto L''(t)$, defined by a UL-Darboux transform followed by a LU-transform, as in Corollary 6.2, with that same choice of σ. It induces the map (6.18) at the level of Toda τ-vectors:*

$$D_n(\rho_t)\longmapsto D_n\left(\rho_t(z)+ce^{\sum_1^\infty t_iz^i}\delta(\lambda-z)\right)=\left(1+c\mathbf{X}(t,\lambda)\right)D_n(\rho_t),\tag{6.24}$$

where $\mathbf{X}(t,\lambda)$ is the Toda lattice vertex operator (6.17).

[14] with asymptotics $\Phi_n(t,\lambda)=e^{\sum t_i\lambda^i}\lambda^n(1+O(\lambda^{-1}))$.

Instead of using Theorems 0.1 and 0.2 to establish those results, one can prove them directly, using the formulae in proposition 6.6 below. In this way, classical formulae have a natural τ-function counterpart.

Proposition 6.6 *Given the weights $\rho_t(z)$, the moments $\mu_i(\rho_t(z))$ and the τ-functions $\tau_n(t) := D_n(\rho_t)$, we have the following expressions for*[15]*:*

- *the monic orthogonal polynomials:*

$$p_n(u;t) = \frac{1}{D_n(\rho_t)} \det \left(\begin{array}{ccc|c} \mu_0 & \dots & \mu_{n-1} & 1 \\ \vdots & & \vdots & \vdots \\ \mu_{n-1} & \dots & \mu_{2n-2} & u^{n-1} \\ \hline \mu_n & \dots & \mu_{2n-1} & u^n \end{array} \right) = \frac{D_n((u-z)\rho_t(z))}{D_n(\rho_t(z))}$$

$$= u^n \frac{\tau_n(t-[u^{-1}])}{\tau_n(t)};$$

$$q_{n-1}(u;t) := \int_{\mathbf{R}^n} \frac{p_n(x;t)}{u-x} \rho_t(x)dx = \frac{1}{D_{n-1}(\rho_t(z))} D_n\left(\frac{\rho_t(z)}{u-z}\right)$$

$$= u^{-n} \frac{\tau_n(t+[u^{-1}])}{\tau_{n-1}(t)}.$$

- *The Christoffel-Darboux kernels: (for h_i, see (6.13))*

$$\sum_{0 \le j \le n} h_j^{-1}(t) p_j(u;t) p_j(v;t) = -\frac{1}{D_{n+1}(\rho_t)} \det \begin{pmatrix} 0 & 1 & v & \dots & v^n \\ 1 & \mu_0 & \mu_1 & \dots & \mu_n \\ u & \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & & & & \\ u^n & \mu_n & \mu_{n+1} & \dots & \mu_{2n} \end{pmatrix}$$

$$= \frac{D_n((u-z)(v-z)\rho_t(z))}{D_{n+1}(\rho_t)}$$

$$= (uv)^n \frac{\tau_n(t-[u^{-1}]-[v^{-1}],\rho)}{\tau_{n+1}(t,\rho)},$$

- *The addition formula:*

$$D_n(\rho_t(z) + c\delta(u-z)) = D_n(\rho_t) + ce^{\sum t_i u^i} D_{n-1}\left((u-z)^2 \rho_t(z)\right)$$

$$= (1 + c\mathbf{X}(t,u))\, D_n(\rho_t).$$

[15] Remember $[\alpha] := (\alpha, \alpha^2/2, \alpha^3/3, \dots)$.

This last identity hinges on the addition formula: For a $n \times n$ moment matrix m_n, the following identity holds:

$$\det\left(m_n(\rho) + c\chi_n(u) \otimes \chi_n(u)\right) = \det m_n(\rho) + c \det m_{n-1}\left((z-u)^2 \rho(z)\right),$$

where

$$\chi_n(u) \otimes \chi_n(v) := \left(u^i v^j\right)_{0 \le i,j \le n}.$$

7 Example 2: "Classical" polynomials, satisfying $2m+1$-step relations

A very natural set of "*classical*" examples is to start from a weight for the standard orthogonal polynomials, thus corresponding to a tridiagonal matrix $\mathcal{L}_1 = \mathcal{L}_2$. Then we perform two consecutive Darboux transforms on the $2m+1$-diagonal matrix $L = \mathcal{L}_1^m = \mathcal{L}_2^m$. This has the effect of mapping a 1-periodic sequence of weights to a generalized m-periodic sequence of weights, thus leading to $2m+1$-band matrices. Therefore, one is lead to a sequence of $2m+1$-step polynomials $\tilde{p}_n^{(1)}$ derived from the "*standard*" ones; they satisfy $2m+1$-step relations, i.e., $z^m \tilde{p}_n^{(1)} = L\tilde{p}_n^{(1)}$, with $2m+1$-diagonal L, but not 3-step relations.

For a general m-periodic weight sequence, for appropriate choices of β and $\tilde{\beta}$, and setting $\lambda = 0$ in (5.2) and (5.8), the compound map

$$L \longmapsto \tilde{L} = (\beta\Lambda^0 + \Lambda)L(\beta\Lambda^0 + \Lambda)^{-1} \longmapsto \tilde{\tilde{L}} = (\Lambda^\top \tilde{\beta} + I)\tilde{L}(\Lambda^\top \tilde{\beta} + I)^{-1} \tag{7.1}$$

induces, according to theorems 0.1, 0.2 and corollary 0.3, the following compound map of weights (assuming $d_{m-1} \neq 0$):

$$\rho \longmapsto \tilde{\rho} = (z\rho_0, z\rho_1, z\rho_2, ...) \longmapsto \tilde{\tilde{\rho}} = \left(\sum_0^{m-1}\left(c_k \delta^{(k)}(z) + d_k \frac{\rho_k(z)}{z^{m-1}}\right), z\rho_0, z\rho_1, ...\right).$$

A particularly interesting case is to start with weights having the form $\rho_k(z) = z^k \rho_0(z)$, where $\rho_0(z)$ is subjected to the following condition:

$$\int_{\mathbf{R}} \left| z^j \rho_0(z) \right| dz < \infty, \quad j \ge -m+1.$$

Then the polynomials $p_n^{(1)}$ are orthogonal with respect to the weight $\rho_0(z)$ and the map above becomes

$$\begin{aligned} \rho = (z^i \rho_0(z))_{0 \le i < \infty} \mapsto \tilde{\tilde{\rho}} &= (\tilde{\tilde{\rho}}_0, \tilde{\tilde{\rho}}_1, \tilde{\tilde{\rho}}_2, \dots) \\ &= \left(\sum_{k=0}^{m-1}\left(c_k \delta^{(k)}(z) + \rho_0(z)\frac{d_{m-k-1}}{z^k}\right), z\rho_0, z^2\rho_0, \dots\right). \end{aligned} \tag{7.2}$$

From the general theory, this new sequence is *generalized m-periodic* with minimal period m. One checks by hand, using $z^m\delta^{(k)}(z)=0$ for $0\le k\le m-1$, that

$$\begin{aligned} z^m\tilde{\tilde{\rho}}_0 &= \sum_{k=0}^{m-1}\Big(c_k z^m\delta^{(k)}(z)+d_{m-k-1}z^{m-k}\rho_0(z)\Big)\\ &= \sum_{k=0}^{m-1} d_{m-k-1}z^{m-k}\rho_0(z)\\ &= \sum_1^m d_{j-1}\tilde{\tilde{\rho}}_j. \end{aligned}$$

The new moments $\tilde{\tilde{\mu}}_{ij}=\langle z^i,\tilde{\tilde{\rho}}_j(z)\rangle$ become:

$$\begin{aligned} \tilde{\tilde{\mu}}_{i0} &= \langle z^i,\tilde{\tilde{\rho}}_0\rangle=\sum_{k=0}^{m-1}\mu_{i-k}d_{m-k-1}+\sum_{k=0}^{m-1}(-1)^k k!c_k\delta_{ik}\\ \tilde{\tilde{\mu}}_{ij} &= \langle z^i,\tilde{\tilde{\rho}}_j\rangle=\langle z^i,z^j\rho_0\rangle=\mu_{i+j}\quad\text{for}\quad j\ge 1, \end{aligned}\tag{7.3}$$

thus defining monic polynomials $\tilde{\tilde{p}}_n^{(1)}(z)$,

$(\det\tilde{\tilde{m}}_n)\ \tilde{\tilde{p}}_n^{(1)}(z)=$

$$\det\begin{pmatrix} \sum_{k=0}^{m-1}\mu_{-k}d_{m-k-1}+\ c_0 & \mu_1 & \mu_2 & \dots & 1\\ \sum_{k=0}^{m-1}\mu_{1-k}d_{m-k-1}-c_1 & \mu_2 & \mu_3 & \dots & z\\ \sum_{k=0}^{m-1}\mu_{2-k}d_{m-k-1}+2!c_2 & \mu_3 & \mu_4 & \dots & z^2\\ \vdots & \vdots & \vdots & \dots & \vdots\\ \sum_{k=0}^{m-1}\mu_{m-k-1}d_{m-k-1}+(-1)^{m-1}(m-1)!c_{m-1} & \mu_m & \mu_{m+1} & \dots & z^{m-1}\\ \sum_{k=0}^{m-1}\mu_{m-k}d_{m-k-1} & \mu_{m+1} & \mu_{m+2} & \dots & z^m\\ \vdots & \vdots & \vdots & \dots & \vdots\\ \sum_{k=0}^{m-1}\mu_{n-k}d_{m-k-1} & \mu_{n+1} & \mu_{n+2} & \dots & z^n \end{pmatrix},$$

which satisfy $2m+1$-step relations

$$z^m p^{(1)}(z)=Lp^{(1)}(z),\quad\text{with a } 2m+1\text{- band matrix } L.$$

Because of the fact that very special cases of these polynomials have appeared in recent work [15] on pentadiagonal matrices, obtained by taking squares of the classical tridiagonal matrices for the Laguerre and Jacobi polynomials, we show how our polynomials can be specialized to those cases. Henceforth, for notational convenience, we replace $\tilde{\tilde{}}$ by $\tilde{}$ in the map (7.2).

Example: **5-step Laguerre polynomials**. Darboux transforms for $L = \mathcal{L}_1^2$, and weight $\rho_0(z) = z^\alpha e^{-z} I_{[0,\infty)}(z)$ for $\alpha > 0$.

Setting $m = 2$ in formule (7.2), we find the map

$$\rho = (\rho_0(z), z\rho_0(z), z^2\rho_0(z), \ldots) \mapsto \tilde\rho = (\tilde\rho_0(z), \tilde\rho_1(z), \tilde\rho_2(z), \ldots),$$

with

$$\tilde\rho_0(z) = \Gamma(\alpha)(c\delta(z) + d\delta'(z)) + (b + \frac{e}{z})\rho_0(z), \quad \text{with } b \neq 0,$$
$$\tilde\rho_i(z) = z^i\rho_0(z) = z^{\alpha+i}e^{-z}I_{[0,\infty)}(z), \quad i \geq 1,$$

obtained from formula (7.2), by setting, for homogeneity considerations and without loss of generality,

$$c_0 = c\Gamma(\alpha), \quad c_1 = d\,\Gamma(\alpha), \quad d_0 = e, \quad d_1 = b.$$

The moments $\langle z^i, \rho_j(z)\rangle$ for the original sequence are given by the following expressions

$$\mu_{ij} = \langle z^i, \rho_j\rangle = \langle z^i, z^j\rho_0\rangle = \Gamma(\alpha + i + j + 1)\ ,$$

with polynomials[16]

$$\begin{aligned} p_n^{(1)}(z) &= \frac{1}{\det m_n}\begin{pmatrix} \alpha! & (\alpha+1)! & (\alpha+2)! & \ldots & 1 \\ (\alpha+1)! & (\alpha+2)! & (\alpha+3)! & \ldots & z \\ (\alpha+2)! & (\alpha+3)! & (\alpha+4)! & \ldots & z^2 \\ (\alpha+3)! & (\alpha+4)! & (\alpha+5)! & \ldots & z^3 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ (\alpha+n)! & (\alpha+n+1)! & (\alpha+n+2)! & \ldots & z^n \end{pmatrix} \\ &= \sum_{i=0}^{n}\binom{n}{i}(\alpha+n)_i(-1)^i z^{n-i}; \end{aligned}$$

the latter are, as expected, the Laguerre polynomials orthogonal with regard to the weight $\rho_0(z)$.

The Darboux transformed moments $\tilde\mu_{ij} = \langle z^i, \tilde\rho_j(z)\rangle$ are given by the following expressions

$$\begin{aligned} \tilde\mu_{i0} &= \langle z^i, \tilde\rho_0\rangle = e\Gamma(\alpha+i) + b\Gamma(\alpha+i+1) + (\delta_{i,0}\, c - \delta_{i,1}\, d)\Gamma(\alpha), \\ \tilde\mu_{ij} &= \langle z^i, \tilde\rho_j\rangle = \langle z^i, z^j\rho_0\rangle = \Gamma(\alpha+i+j+1) \quad \text{for} \quad j \geq 1, \end{aligned}$$

from which one computes the Darboux transformed monic polynomials

[16] $\alpha! := \Gamma(\alpha+1)$, $(\alpha)_0 = 1$ and $(\alpha)_j = \alpha(\alpha-1)\ldots(\alpha-j+1)$.

$(\det \tilde m_n)\ \tilde p_n^{(1)}(z) =$

$$\left(\begin{array}{ccccc} (\alpha-1)!\,e+\alpha!\,b+(\alpha-1)!\,c & (\alpha+1)! & (\alpha+2)! & \dots & 1 \\ \alpha!\,e+(\alpha+1)!\,b-(\alpha-1)!\,d & (\alpha+2)! & (\alpha+3)! & \dots & z \\ (\alpha+1)!\,e+(\alpha+2)!\,b & (\alpha+3)! & (\alpha+4)! & \dots & z^2 \\ (\alpha+2)!\,e+(\alpha+3)!\,b & (\alpha+4)! & (a+5)! & \dots & z^3 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ (\alpha+n-1)!\,e+(\alpha+n)!\,b & (\alpha+n+1)! & (\alpha+n+2)! & \dots & z^n \end{array}\right). \quad (7.4)$$

The appendix to this paper gives the first four 5-step Laguerre polynomials.

The classical Laguerre polynomials are evidently special cases of the Darboux transformed polynomials $\tilde p_n^{(1)}$'s:

$$p_n^{(1)}(z) = \tilde p_n^{(1)}(z)|_{c=d=e=0,\ b=1}.$$

It is interesting that, in an effort to find bispectral problems, Grünbaum and Haine [15] had obtained special cases of these polynomials. Their method was to perform two explicit Darboux transforms on the explicit square $L = \mathcal{L}^2$ of the 3-step relation $\mathcal{L}$ for the Laguerre polynomials. They found, by computation, a new matrix $\tilde L$ and polynomials $\tilde p(z)$, which coincide with ours, by setting $c = d = 0,\ e/b = \alpha/r$ in (7.4), and hence $r \neq 0$. They show they are related to Laguerre by means of a differential equation. Indeed, given the differential equation for the Laguerre polynomials,

$$B = -z\frac{\partial^2}{\partial z^2} + (z-\alpha-1)\frac{\partial}{\partial z}, \quad \text{with} \quad Bp_n(z) = np_n(z),$$

and the operators

$$P = B + \frac{\partial}{\partial z} + r \quad \text{and} \quad Q = B - \frac{\partial}{\partial z} + r + 1,$$

they show that the $p_n^{(1)}$'s and $\tilde p_n^{(1)}$'s are related by the following differential equations

$$Pp_n(z) = (n+r)\tilde p_n(z) \quad \text{and} \quad Q\tilde p_n(z) = (n+r+1)p_n(z).$$

Example: **5-step Jacobi polynomials**. Darboux transform for $L = \mathcal{L}^2$ and Jacobi weight[17] $\rho_0(z) = (2-z)^\alpha z^\beta I_{[0,2]}(z)$, for $\alpha > -1$ and $\beta > 0$. Here the map is given by $\rho \longmapsto \tilde\rho$, with

$$\begin{aligned} \tilde\rho_0(z) &= \nu\,(c\ \delta(z) + d\ \delta'(z)) + \rho_0(z)\left(e + \frac{b}{z}\right), \quad \text{with} \quad e \neq 0 \\ \tilde\rho_i(z) &= z^i\rho_0(z) = (2-z)^\alpha z^{\beta+i} I_{[0,2]}(z) \quad \text{for} \quad i \geq 1, \end{aligned} \qquad (7.5)$$

[17]It is more convenient to base the Jacobi weight on [0,2] rather then [-1,1].

with

$$\nu = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

As in the previous example, the adjustments of constants was made for homogeneity reasons.

The moments for the original sequence are given by

$$\mu_{ij} = \langle z^i, \tilde{\rho}_j\rangle = 2^{\alpha+\beta+i+j+1}\frac{\alpha!(\beta+i+j)!}{(\alpha+\beta+i+j+1)!} \quad \text{for} \quad j \geq 1,$$

and the Jacobi polynomials by

$$p_n^{(1)}(z) = \frac{1}{\det m_n}\times$$

$$\det\begin{pmatrix} \frac{\alpha!\,2^{\beta+\alpha+1}\,\beta!}{(\beta+\alpha+1)!} & \frac{\alpha!\,2^{\beta+\alpha+2}\,(\beta+1)!}{(\beta+\alpha+2)!} & \frac{\alpha!\,2^{\beta+\alpha+3}\,(\beta+2)!}{(\beta+\alpha+3)!} & \ldots & 1 \\ \frac{\alpha!\,2^{\beta+\alpha+2}\,(\beta+1)!}{(\beta+\alpha+2)!} & \frac{\alpha!\,2^{\beta+\alpha+3}\,(\beta+2)!}{(\beta+\alpha+3)!} & \frac{\alpha!\,2^{\beta+\alpha+4}\,(\beta+3)!}{(\beta+\alpha+4)!} & \ldots & z \\ \frac{\alpha!\,2^{\beta+\alpha+3}\,(\beta+2)!}{(\beta+\alpha+3)!} & \frac{\alpha!\,2^{\beta+\alpha+4}\,(\beta+3)!}{(\beta+\alpha+4)!} & \frac{\alpha!\,2^{\beta+\alpha+5}\,(\beta+4)!}{(\beta+\alpha+5)!} & \ldots & z^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\alpha!\,2^{\beta+\alpha+n+1}\,(\beta+n)!}{(\beta+\alpha+n+1)!} & \frac{\alpha!\,2^{\beta+\alpha+n+2}\,(\beta+n+1)!}{(\beta+\alpha+n+2)!} & \frac{\alpha!\,2^{\beta+\alpha+n+3}\,(\beta+n+2)!}{(\beta+\alpha+n+3)!} & \ldots & z^n \end{pmatrix}$$

$$= \frac{1}{\det m_n}\sum_{k=0}^{n}(-2)^{n-k}\binom{n}{k}(\alpha+\beta+n+k)_k(\beta+n)_{n-k}z^k.$$

The Darboux transformed moments are given by

$$\langle z^i, \tilde{\rho}_0\rangle = 2^{\alpha+\beta+i+1}\frac{\alpha!(\beta+i)!}{(\alpha+\beta+i+1)!}\left((e+c\delta_{i0}) + (b-d\delta_{i1})\frac{\alpha+\beta+i+1}{2(\beta+i)}\right)$$

$$\langle z^i, \tilde{\rho}_j\rangle = 2^{\alpha+\beta+i+j+1}\frac{\alpha!(\beta+i+j)!}{(\alpha+\beta+i+j+1)!} \quad \text{for} \quad j \geq 1,$$

and the new polynomials $\tilde{p}_n^{(1)}$ by:

$$\tilde{p}_n^{(1)} = \frac{1}{\det m_n}\times\det$$

$$\begin{pmatrix} \frac{\alpha!\,2^{\beta+\alpha+1}\,\beta!\left(e+c+\frac{b\,(\beta+\alpha+1)}{2\,\beta}\right)}{(\beta+\alpha+1)!} & \frac{\alpha!\,2^{\beta+\alpha+2}\,(\beta+1)!}{(\beta+\alpha+2)!} & \frac{\alpha!\,2^{\beta+\alpha+3}\,(\beta+2)!}{(\beta+\alpha+3)!} & \ldots & 1 \\ \frac{\alpha!\,2^{\beta+\alpha+2}\,(\beta+1)!\left(e+\frac{(\beta+\alpha+2)\,(b-d)}{2\,(\beta+1)}\right)}{(\beta+\alpha+2)!} & \frac{\alpha!\,2^{\beta+\alpha+3}\,(\beta+2)!}{(\beta+\alpha+3)!} & \frac{\alpha!\,2^{\beta+\alpha+4}\,(\beta+3)!}{(\beta+\alpha+4)!} & \ldots & z \\ \frac{\alpha!\,2^{\beta+\alpha+3}\,(\beta+2)!\left(e+\frac{b\,(\beta+\alpha+3)}{2\,(\beta+2)}\right)}{(\beta+\alpha+3)!} & \frac{\alpha!\,2^{\beta+\alpha+4}\,(\beta+3)!}{(\beta+\alpha+4)!} & \frac{\alpha!\,2^{\beta+\alpha+5}\,(\beta+4)!}{(\beta+\alpha+5)!} & \ldots & z^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\alpha!\,2^{\beta+\alpha+n+1}\,(\beta+n)!\left(e+\frac{b\,(\beta+\alpha+n+1)}{2\,(\beta+n)}\right)}{(\beta+\alpha+n+1)!} & \frac{\alpha!\,2^{\beta+\alpha+n+2}\,(\beta+n+1)!}{(\beta+\alpha+n+2)!} & \frac{\alpha!\,2^{\beta+\alpha+n+3}\,(\beta+n+2)!}{(\beta+\alpha+n+3)!} & \ldots & z^n \end{pmatrix}.$$

Again in [15], Grünbaum and Haine had considered special cases of these polynomials. Namely, the Jacobi polynomials satisfy a differential equation,

$$Bp_n^{(1)} = n(n+\alpha+\beta+1)p_n^{(1)},$$

involving the differential operator

$$B = z(z-2)\left(\frac{\partial}{\partial z}\right)^2 + ((\alpha+\beta+2)z - 2(\beta+1))\frac{\partial}{\partial z}.$$

Defining

$$P = B - (z-2)\frac{\partial}{\partial z} + r \quad \text{and} \quad Q = B + (z-2)\frac{\partial}{\partial z} + r + \alpha + \beta + 1,$$

they show the $p_n^{(1)}$ and $\tilde{p}_n^{(1)}$'s, for $c=0, d=0, e/b = r/2\beta$ and hence $r \neq 0$, are related by the following differential equations:

$$Pp_n^{(1)} = (n^2 + (\alpha+\beta)n + r)\tilde{p}_n^{(1)}$$

$$Q\tilde{p}_n^{(1)} = (n^2 + (\alpha+\beta+2)n + \alpha+\beta+r+1)p_n^{(1)}.$$

This paper shows that these polynomials have a determinantal representation in terms of moments, defined with respect to periodic sequences of weights. Moreover, the vertex operator technology enables one to consider general $2m+1$-band matrices. It remains an interesting open question to investigate the differential equations satisfied by the general $2m+1$-step Laguerre or Jacobi polynomials.

8 Appendix

The first few 5-step Laguerre polynomials are given by the following polynomials, which, for convenience of notation, we did not make monic; set $\alpha = a$:

$$\begin{aligned}
\tilde{p}_1^{(1)}(z) &= (e+c+a\,b)\,z - a\,e + d - a^2\,b - a\,b \\
\tilde{p}_2^{(1)}(z) &= (2\,e+d+a\,c+2\,c+a\,b)\,z^2 \\
&\quad - \left(4\,a\,e+6\,e+a^2\,c+5\,a\,c+6\,c+2\,a^2\,b+4\,a\,b\right)\,z \\
&\quad + (a+2)\,\left(2\,a\,e-a\,d-3\,d+a^2\,b+a\,b\right)
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_3^{(1)}(z) &= \left(6\,e+2\,a\,d+6\,d+a^2\,c+5\,a\,c+6\,c+2\,a\,b\right)\,z^3 \\
&\quad -(18\,a\,e+48\,e+3\,a^2\,d+21\,a\,d+36\,d+2\,a^3\,c+18\,a^2\,c+ \\
&\qquad\qquad 52\,a\,c+48\,c+6\,a^2\,b+18\,a\,b)\,z^2 \\
&\quad + (a+3)\,\left(18\,a\,e+24\,e+a^3\,c+9\,a^2\,c+26\,a\,c+24\,c+6\,a^2\,b+12\,a\,b\right)\,z
\end{aligned}$$

$$-(a+2)\,(a+3)\,\Big(6\,a\,e-a^2\,d-7\,a\,d-12\,d+2\,a^2\,b+2\,a\,b\Big)$$

$$\begin{aligned}
\tilde{p}_4^{(1)}(z) \;=\;& \Big(24\,e+3\,a^2\,d+21\,a\,d+36\,d+a^3\,c+9\,a^2\,c+26\,a\,c+24\,c+6\,a\,b\Big)\,z^4\\
&-(96\,a\,e+360\,e+8\,a^3\,d+96\,a^2\,d+376\,a\,d+480\,d+3\,a^4\,c\\
&\qquad +42\,a^3\,c+213\,a^2\,c+462\,a\,c+360\,c+24\,a^2\,b+96\,a\,b)\,z^3\\
&+3\,(a+4)\,\Big(48\,a\,e+120\,e+2\,a^3\,d+24\,a^2\,d+94\,a\,d+120\,d+a^4\,c\\
&\qquad +14\,a^3\,c+71\,a^2\,c+154\,a\,c+120\,c+12\,a^2\,b+36\,a\,b\Big)\,z^2\\
&-(a+3)\,(a+4)\,\Big(96\,a\,e+120\,e+a^4\,c+14\,a^3\,c+71\,a^2\,c+154\,a\,c\\
&\qquad +120\,c+24\,a^2\,b+48\,a\,b\Big)\,z\\
&+(a+2)\,(a+3)\,(a+4)\,\Big(24ae-a^3d-12a^2d-47ad-60d+6a^2b+6ab\Big),
\end{aligned}$$

etc...

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